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(Continued on the inside back cover)

ON THE PROBLEM OF THREE BODIES IN A PLANE

by

Victor G. Szebehely*

INTRODUCTION

In this paper the 12th order system of differential equations of the planar three bodies problem is reduced to one of the 4th order by introducing generalized coordinates.

The system of differential equations of the motion for the planar problem is one of the 12th order, because the positions of the three bodies require 6 coordinates and the kinematic equations are of the second order. This system is reducible from the 12th to the 8th order by using four integrals regarding the motion of the center of gravity. By means of a contact transformation, this can be reduced to the 6th order and using the integral of energy and eliminating the time, a system of 4th order can be obtained as the final result.

In the present paper, a new set of generalized coordinates is introduced by means of which the final form of the 4th order system can be obtained without a contact transformation. The special case for which the masses of the three bodies are equal and the attracting forces are proportional to the cubes of the distances is investigated in detail and two sets of particular solutions are obtained. One of these is represented by trochoidal curves, and the other set splits into two, each of which can be considered as a generalization of the Lagrangian collinear solution.

GENERALIZED COORDINATES

In two dimensions the positions of the three bodies are described by 6 generalized coordinates. The following coordinate systems are introduced:

(1) the (ξ,η) system which is fixed in the plane of motion,

(2) the (x,y) system which has its origin at the center of gravity of the three bodies, the coordinates of which are ξ_0 and η_0. The x and y axes coincide with the principal axes of inertia of the three bodies. The (x,y) system is moving with a translational velocity $\dot{\vec{\rho}}_0(\dot{\xi}_0,\dot{\eta}_0)$ and it rotates around its origin with the angular velocity $\dot{\psi}$, where ψ is the angle from the ξ to the x axis.

The motion of the center of gravity is described by $\ddot{\xi}_0 = \ddot{\eta}_0 = 0$. Now that the motion of the centroid of the system has been determined,

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the following four generalized coordinates are introduced:[1]

(1) the first radius of gyration,

$$q_1 = \sqrt{\frac{1}{M} \Sigma\, m_i x_i^{\,2}}, \quad \text{where } M = \sum_{i=1}^{3} m_i$$

(2) the second radius of gyration,

$$q_2 = \sqrt{\frac{1}{M} \Sigma\, m_i y_i^{\,2}}, \quad \text{sign } (q_1 q_2) = \text{sign } |1, x_i, y_i|,$$

(3) the generalized angular coordinate,

$$\varkappa = \text{arc tan } \frac{y_i}{x_i} \sqrt{\frac{\Sigma\, m_i x_i^{\,2}}{\Sigma\, m_i y_i^{\,2}}} - \delta_i,$$

where the constants δ_i satisfy the equation, $\tan(\delta_i - \delta_j) = \sqrt{\frac{M m_k}{m_i m_j}}$ and (4) the angle of rotation, ψ, which has already been introduced.

The (x_i, y_i) coordinates are related to the q_1, q_2, $\varkappa$ generalized coordinates as follows:

$$x_i = q_1 \sqrt{\frac{m_j + m_k}{m_i}} \cos(\varkappa + \delta_i); \quad y_i = q_2 \sqrt{\frac{m_j + m_k}{m_i}} \sin(\varkappa + \delta_i) \ldots \qquad (1)$$

$$(i, j, k = 1, 2, 3)$$

EQUATIONS OF MOTION

The use of the generalized coordinates leads to the following expression of the kinetic energy of the system:

$$T = \frac{M}{2}[\dot{\xi}_0^2 + \dot{\eta}_0^2 + (q_1^2 + q_2^2)(\dot{\varkappa}^2 + \dot{\psi}^2) + 4q_1 q_2 \dot{\varkappa}\dot{\psi} + \dot{q}_1^2 + \dot{q}_2^2] \ldots \qquad (2)$$

The potential energy has the form:

$$V = KM^{\frac{n}{2}} \Sigma\, m_i m_j \left[\frac{1}{m_i} + \frac{1}{m_j}\right]^{\frac{n}{2}} [q_1^2 \sin^2(\varkappa + \delta_k) + q_2^2 \cos^2(\varkappa + \delta_k)]^{\frac{n}{2}} \ldots \qquad (3)$$

Here K is a constant and the forces are proportional to the $(n - 1)$st. power of the distances. Formulas (2) and (3) are obtained by elementary, straight forward calculations.

The Lagrangian equations of motion are

[1]Radau, R. Comptes Rendus, 68, 1465-1468.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where $L = T - V$ is the kinetic potential,

q_i, the generalized position coordinates (in our case q_1, q_2, $\varkappa$ and ψ)

$\dot{q}_i$, the generalized velocity coordinates (in our case $\dot{q}_1$, $\dot{q}_2$, $\dot{\varkappa}$ and $\dot{\psi}$).

Due to the fact that V is independent of the generalized velocity coordinates, the equations of motion become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \tag{4}$$

$$\left.\begin{aligned}
&\text{For } q_i = q_1, \text{ this is } && \ddot{q}_1 - (\dot{\varkappa}^2 + \dot{\psi}^2)q_1 - 2q_2\dot{\varkappa}\dot{\psi} = -\frac{1}{M}\frac{\partial V}{\partial q_1} \\
&\text{for } q_i = q_2, && \ddot{q}_2 - (\dot{\varkappa}^2 + \dot{\psi}^2)q_2 - 2q_1\dot{\varkappa}\dot{\psi} = -\frac{1}{M}\frac{\partial V}{\partial q_2} \\
&\text{for } q_i = \varkappa && \frac{d}{dt}\left[(q_1^2 + q_2^2)\dot{\varkappa} + 2q_1q_2\dot{\psi}\right] = -\frac{1}{M}\frac{\partial V}{\partial \varkappa}
\end{aligned}\right\} \tag{5}$$

$$\text{for } q_i = \psi \qquad \frac{d}{dt}\left[(q_1^2 + q_2^2)\dot{\psi} + 2q_1q_2\dot{\varkappa}\right] = 0 \tag{6}$$

Equation (6) is the result of the fact that ψ is an ignorable (cyclic) coordinate. From equation (6) can be obtained the integral

$$(q_1^2 + q_2^2)\dot{\psi} + 2q_1q_2\dot{\varkappa} = 0$$

which corresponds to the integral of angular momentum. Using in addition to this, the integral of energy, the equations of motion can be reduced to the 4th. order.

SPECIAL CASES

Using the system of equations (5), several well known particular solutions can be obtained such as Lagrange's collinear solution, Lagrange's equidistant solution, O. Pylarinos'[1] homothetic motion, D. Sokolov's[2] particular solution, etc.

It has already been mentioned that the coordinate ψ is cyclic. The $\varkappa$ coordinate is cyclic as far as the kinetic energy is concerned.

[1] O. Pylarinos: "Ueber die Lagrangeschen Faelle in verallgemeinerten Dreikoerperproblem." Mathematische Zeitschrift, Vol. 47, pp. 351-372.

[2] G. Sokolov: "Sur un nouveau cas d'intégrabilité dans le problème rectiligne de trois corps." C. R. (Doklady) Acad. Sci. URSS (N.S.) Vol. 46, pp. 95-98.

It can be shown that it is also cyclic with respect to the potential energy if the masses are equal and the forces vary as the cubes of the distances.

For this case

$$x_i = \sqrt{2}\, q_1 \cos(\varkappa + \frac{2\pi}{3} i); \quad y_i = \sqrt{2}\, q_2 \sin(\varkappa + \frac{2\pi}{3} i);$$

$$T = 3m/2[\dot{\xi}_0^2 + \dot{\eta}_0^2 + (q_1^2 + q_2^2)(\dot{\varkappa}^2 + \dot{\psi}^2) + 4q_1 q_2 \dot{\varkappa}\dot{\psi} + \dot{q}_1^2 + \dot{q}_2^2] \quad (7)$$

$$V = \frac{27\ Km^2}{2} (3q_1{}^4 + 2q_1{}^2 q_2{}^2 + 3q_2{}^4) \quad (8)$$

Since the $\varkappa$ and the ψ coordinates are cyclic,

$$\frac{\partial T}{\partial \dot{\psi}} = \text{const.} \quad \text{and} \quad \frac{\partial T}{\partial \dot{\varkappa}} = \text{const.}$$

The last formula for the kinetic energy suggests the introduction of four new generalized coordinates:

$$\left.\begin{aligned} \Phi_1 &= q_1 + q_2 & \Phi_2 &= q_1 - q_2 \\ \Phi_3 &= \psi + \varkappa & \Phi_4 &= \psi - \varkappa \end{aligned}\right\} \quad (9)$$

In terms of these new generalized coordinates the kinetic energy, the potential energy and the equations of motion are

$$T = \frac{3m}{4} [2(\dot{\xi}_0^2 + \dot{\eta}_0^2) + \Phi_1^2 \dot{\Phi}_3^2 + \Phi_2^2 \dot{\Phi}_4^2 + \dot{\Phi}_1^2 + \dot{\Phi}_2^2] \quad (10)$$

$$V = \frac{27\ Km^2}{4} (\Phi_1^4 + 4\Phi_1^2 \Phi_2^2 + \Phi_2^4) \quad (11)$$

$$\left.\begin{aligned} \ddot{\Phi}_1 - \Phi_1 \dot{\Phi}_3^2 &= -18\ Km\Phi_1(\Phi_1^2 + 2\Phi_2^2) \\ \ddot{\Phi}_2 - \Phi_2 \dot{\Phi}_4^2 &= -18\ Km\Phi_2(\Phi_2^2 + 2\Phi_1^2) \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} \dot{\Phi}_3 \Phi_1^2 &= c_1 \\ \dot{\Phi}_4 \Phi_2^2 &= c_2 \end{aligned}\right\} \quad (13)$$

Eliminating $\dot{\Phi}_3$ and $\dot{\Phi}_4$ from (12) by means of (13) and using convenient units ($18\ Km = 1$) the final form of the fourth order system of differential equations is obtained:

$$\left.\begin{aligned}\ddot{\Phi}_1 - \frac{c_1^2}{\Phi_1^3} + \Phi_1(\Phi_1^2 + 2\Phi_2^2) &= 0\\ \ddot{\Phi}_2 - \frac{c_2^2}{\Phi_2^3} + \Phi_2(\Phi_2^2 + 2\Phi_1^2) &= 0\end{aligned}\right\} \tag{14}$$

PARTICULAR SOLUTIONS

(1) The above equations are satisfied with constant Φ_1 and Φ_2 for which case

$$x_i = \frac{\Phi_1 + \Phi_2}{\sqrt{2}} \cos(\kappa + \frac{2\pi}{3} i), \quad y_i = \frac{\Phi_1 - \Phi_2}{\sqrt{2}} \sin(\kappa + \frac{2\pi}{3} i) \tag{15}$$

The positions of the particles in the fixed (ξ,η) system are obtained by considering the translation and rotation of the (x,y) coordinate system with respect to the (ξ,η) system.

$$\left.\begin{aligned}\xi_i &= a_\xi(t - t_0) + b_\xi + x_i \cos\psi - y_i \sin\psi\\ \eta_i &= a_\eta(t - t_0) + b_\eta + x_i \sin\psi + y_i \cos\psi\end{aligned}\right\} \tag{16}$$

It can be assumed without any loss of generality that the initial position of the center of gravity coincides with the origin of the fixed coordinate system, in which case

$$b_\xi = b_\eta = 0$$

and that the velocity of the center of gravity is zero, in which case

$$a_\xi = a_\eta = 0.$$

Introducing into formulas (16) the expressions obtained in (15) for x_i and y_i gives

$$\left.\begin{aligned}\xi_i &= \frac{\Phi_1}{\sqrt{2}} \cos(\Phi_3 + \frac{2\pi}{3} i) + \frac{\Phi_2}{\sqrt{2}} \cos(\Phi_4 - \frac{2\pi}{3} i)\\ \eta_i &= \frac{\Phi_1}{\sqrt{2}} \sin(\Phi_3 + \frac{2\pi}{3} i) + \frac{\Phi_2}{\sqrt{2}} \sin(\Phi_4 - \frac{2\pi}{3} i)\end{aligned}\right\} \tag{17}$$

Making use of the fact that $\dot{\Phi}_3$ and $\dot{\Phi}_4$ are constants and choosing proper initial values, the complex position vector for the j-th particle becomes

$$\zeta_j = \xi_j + i\eta_j = Ae^{i(\alpha t + 2\pi/3 j)} + Be^{i(\beta t - 2\pi/3 j)} \tag{18}$$

where

$$\begin{aligned} & i = \sqrt{-1}, \quad j = 1,2,3, \\ & \left.\begin{aligned} \alpha^2 &= 2(A^2 + 2B^2) \\ \beta^2 &= 2(B^2 + 2A^2) \end{aligned}\right\} \end{aligned} \tag{19}$$

Equation (18) represents a trochoidal curve.
Two theorems can be established regarding this solution,

(a) The area of the triangle formed by the three bodies is $\frac{3\sqrt{3}}{2} q_1 q_2$ = const.

(b) The sum of the squares of the sides of the triangle is $9(q_1^2 + q_2^2)$ = const.

According to (b) the distance between the bodies is limited, and since in addition by (a) the area is constant (and except for special cases different from zero), it follows that the bodies cannot collide. Therefore, the motion by Laplace's definition is stable. Theorems (a) and (b) also suggest the use of another set of generalized coordinates, $q_1 \cdot q_2$ and $q_1^2 + q_2^2$.

A further analysis of equations (18) and (19) results in a one-parametric system of epi-and hypocycloidal curves as orbits. Special cases correspond to the Lagrangian collinear and equidistant solutions. A special case of the epicycloidal solution is obtained when the three bodies lie in fixed positions on a straight line, which is rotating with constant angular velocity. The degenerate case of the hypocycloidal solution is a fixed straight line, on which the bodies are moving. The equidistant solution can be obtained by letting the hypo-and epicycloidal orbits coincide. In this case the equilateral triangle formed by the three bodies rotates uniformly. The above statements will be verified in another paper.

(2) Two further more general collinear solutions can be obtained by putting $q_2 = 0$. Then $\Phi_1 = \Phi_2 = \Phi$ and equations (20) reduce to

$$\ddot{\Phi}\Phi^3 + 3\Phi^6 = c^2 \tag{20}$$

where

$$c^2 = c_1^2 = c_2^2 \tag{21}$$

The solution of (20) can be obtained in terms of an elliptic integral:

$$t = t_0 + \int_{\Phi_0}^{\Phi} \frac{\Phi \, d\Phi}{[(3/2\Phi_0^4 + \dot{\Phi}_0^2 + c^2/\Phi_0^2)\Phi^2 - 3/2\Phi^6 - c^2]^{\frac{1}{2}}} = F(\Phi),$$

whose inverse is $\Phi = f(t)$.

Since $q_2 = 0$, the three bodies are located on the rotating x axis and

$$x_i = \sqrt{2}\,\Phi \cos(\varkappa + \frac{2\pi}{3} i), \quad y_i \equiv 0$$

In consequence of equation (21)

(a) $c_1 = -c_2$ or

(b) $c_1 = c_2$

Analyzing cases (a) and (b) two new generalized collinear solutions can be obtained.

For case (a), according to equations (13) it follows that

$$\Phi^2(\dot{\varphi}_3 + \dot{\varphi}_4) = 0$$

or

$$\Phi^2 \dot{\psi} = 0$$

If the trivial solution $\Phi = 0$ is disregarded, then $\dot{\psi} = 0$ and therefore $\psi = \psi_0 = $ const. A repeated use of equations (13) gives

$$\dot{\varkappa} = \frac{c_1}{\Phi^2}; \quad \varkappa = \int_{t_0}^{t} \frac{c_1}{\Phi^2}\, dt = c_1 \int_{t_0}^{t} \frac{dt}{f^2(t)}$$

The solution in the moving coordinate system is

$$x_i = \sqrt{2}\, f(t) \cos[c_1 \int_{t_0}^{t} \frac{dt}{f^2(t)} + \frac{2\pi}{3} i]; \quad y_i \equiv 0$$

and in the fixed system,

$$\left.\begin{aligned} \xi_i &= \sqrt{2}\, f(t) \cos \psi_0 \cos[c_1 \int_{t_0}^{t} \frac{dt}{f^2(t)} + \frac{2\pi}{3} i] \\ \eta_i &= \sqrt{2}\, f(t) \sin \psi_0 \cos[c_1 \int_{t_0}^{t} \frac{dt}{f^2(t)} + \frac{2\pi}{3} i] \end{aligned}\right\} \qquad (22)$$

Introducing the polar coordinates

$$\rho_i = \sqrt{\xi_i^2 + \eta_i^2} \quad \text{and} \quad \alpha_i = \arctan \frac{\eta_i}{\xi_i}$$

equations (22) become

$$\left.\begin{aligned} \rho_i &= \sqrt{2}\, f(t)\cos\left[c_1\int_{t_0}^{t}\frac{dt}{f^2(t)} + \frac{2\pi}{3}\, i\right] \\ \text{and} \qquad \alpha_i &= \psi_0 \end{aligned}\right\} \tag{23}$$

In the above solution the three bodies are moving along a fixed straight line which makes an angle ψ_0 with the ξ axis, and the motion is described by formula (23).

In case (b) the solution is

$$\left.\begin{aligned} \rho_i &= \sqrt{2}\, f(t)\cos\left(\kappa_0 + \frac{2\pi}{3} i\right) \\ \alpha_i &= c_1\int_{t_0}^{t}\frac{dt}{f^2(t)} \end{aligned}\right\} \tag{24}$$

In this solution the bodies are moving on a rotating straight line and are governed by the $f(t)$ function. The variable angular velocity of the rotating line is given by $c_1/f^2(t)$.

Case (b) can be considered a very general case of collinear motion.

Regarding the generalization of the above expanded method, the three dimensional four bodies problem can be mentioned. When the above introduced Lagrangian coordinates are used in a more general sense, then the 24th. order system of differential equations of the three dimensional four bodies problem is reducible to the 12th. order. The reduction is again symmetric and does not require any contact transformations. The three dimensional three bodies problem, being a degenerate case of the four bodies problem, can be of course investigated also with the above given method, however no complete symmetry can be expected.

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THEORY OF BUDGETS BASED ON PARABOLIC ENGEL CURVES*

G. A. Baker

1. *Introduction.* In econometrics it is usual to assume that the quantities of commodities purchased depend only upon the total income. Thus if x is the total income then the expenditure for the ith item is represented by $x_i(x)$. Suppose that there are n items. Now if x is regarded as a parameter then the functions x_i will vary as x assumes different values and will trace a curve in the n-dimensional quantity space $(x_1, x_2, \ldots x_n)$. This curve is called an Engel Curve after Ernest Engel, a German economist. The Engel Curves can be determined empirically by observing the consumption of individuals belonging to different income classes.

It has been generally assumed that the expenditure on a single item expressed as a function of total income is linear [1], [2], [3]. Allen and Bowley recognize that the range must be much restricted for the linear relationship to hold. Wald simply assumes that the linear relationship holds in some sufficiently restricted region but does not investigate the length of the interval in which such an assumption is valid.

Davis discusses only the case for linear curves but suggests an equation of the form

$$x_i = a - be^{-\mu x} \tag{1.1}$$

where x_i is the expenditure on a particular item; x is total income; a, b, and μ are positive numbers with $b < a$, as being suitable over a more extended range. The difficulties with this suggestion are that the curve to represent the expenditure on the combination of two items is not the combination of the representations of the two separate items and that an asymptotic limit is assumed which may well be questioned. It is true that only a limited amount of food can be consumed but the money value of this food may increase indefinitely.

It must be realized that categories such as food, clothing, etc. are more or less arbitrary and in reality include many more or less related but distinct items. Thus in fitting a curve to expenditures on food, we are really finding a curve that is the resultant of the combination of many more particularized expenditure curves.

* Presented at the fourth annual meeting of the Northern California Section of the American Mathematical Association, January 31, 1942, University of California at Berkeley.

It is very important that the mathematical models of the economic world be as realistic and as extended in scope as possible. The purpose of this paper is to give a more realistic and extended basis for the theory of budgets.

Lately a great deal of budgetary data has become available as a result of the Consumer Purchases Study carried on as a joint project by the Bureau of Home Economics and Bureau of Labor Statistics with the cooperation of the National Resources Planning Board, the Works Progress Administration, and the Central Statistical Board.

An extensive examination of these data indicate that a least-square parabola of second degree will adequately represent the functional relation between expenditure on an item and total income or expenditure on all items for a very extended range of the income variable. Parabolas have the advantage that the representations for the separate items can be added for the representation of the total of any subset of items or the total of all of the separate items. If the functional relation of expenditure on an item to total expenditure is parabolic then the Engel Curve as defined by Wald, loc. cit., page 145, equations (2) is parabolic for two expenditures. If only two expenditures are considered and the Engel Curve is parabolic and the utility or indicator function quadratic, then the utility function is determined except for a constant multiplier and an additive constant. The degree of determinateness is much less in the case of n items since the utility function then involves an arbitrary positive definite quadratic form of $n - 2$ variables.

2. *Engel curves and utility function.* Let us consider two expenditures, x_1 = expenditure on savings and x_2 = expenditure on all other items. Then $x = x_1 + x_2$ is the total income or expenditure. For a particular set of data [4], given in detail in table 1 we have

$$(2.1) \qquad \begin{aligned} x_1 &= .0000545\,x^2 + .0433\,x - 61.3 \\ x_1 &= -.0000545\,x^2 + .9567\,x + 61.3 \end{aligned}$$

Eliminating x between the equations of (2.1) we obtain

$$(2.2) \qquad \begin{aligned} (x_1 + x_2)^2 + 7.9450 \times 10^2(x_1 + x_2) - 1.8349 \times 10^4 x_1 \\ -1.1248 \times 10^6 = 0 \end{aligned}$$

as the Engel Curve, which is a parabola. It should be noted that the Engel Curve can be parabolic only if the prices paid for essentially the same items differ at different income levels.

In Davis' (loc. cit. pp. 166-168) notation the parabolic Engel Curve is given by

$$(2.3) \qquad 2\rho_2(l_1 x_1 + m_1 x_2) + r_1 = K[\rho_2(l_2 x_1 + m_2 x_2) + r_2]^2$$

TABLE 1

Expenditure Patterns of Southeast Village Families Containing only Husband and Wife.

Average Total Income	No. of families	Expenditure on savings Actual	Expenditure on savings Computed	Expenditure on all other items Actual	Expenditure on all other items Computed
404	18	-3	-34	407	439
579	54	-6	-18	585	597
788	63	-3	6	791	781
1047	55	58	44	989	1003
1303	69	64	88	1239	1215
1508	67	117	128	1391	1380
1743	38	195	180	1548	1563
2069	50	268	262	1801	1807
2511	19	333	391	2178	2120
3087	17	653	592	2434	2495
4355	6	1212	1162	3143	3193
7244	7	3092	3114	4152	4130

if the utility function is

$$(2.4)\quad U(x_1,\ x_2) = Ax_1^2 + 2Bx_1x_2 + Cx_2^2 + 2Dx_1 + 2Ex_2 + F.$$

Take $l_1 = l_2 = 1$ and we find that

$$(2.5)\quad \begin{aligned} K &= -4.7525 \times 10^{-8} \\ \rho_2 &= -4.5871 \times 10^3 \\ r_2 &= 1.9221 \times 10^7 \\ r_1 &= -1.8683 \times 10^7 \end{aligned}$$

The values of the coefficients of (2.4) are then determined as

$$(2.6)\quad \begin{aligned} A &= -3.4403 \times 10^3 & D &= 1.3450 \times 10^5 \\ B &= 1.1468 \times 10^3 & E &= 9.4760 \times 10^6 \\ C &= -3.4403 \times 10^3 & F &= \text{arbitrary} \end{aligned}$$

Hence a utility function is

$$U(x_1, x_2) = -3.4403 \times 10^3 x_1^2 + 2.2936 \times 10^3 x_1 x_2 -3.4403 \times 10^3 x_2^2 + 2.6900 \times 10^5 x_1 + 1.8952 \times 10^7 x_2 + F. \tag{2.7}$$

$U_{11} = -6.8806 \times 10^3 < 0$ and

$$\begin{vmatrix} U_{11} U_{12} \\ U_{21} U_{22} \end{vmatrix} = \begin{vmatrix} -6.8806 \times 10^3 & -2.2936 \times 10^3 \\ -2.2936 \times 10^3 & -6.8806 \times 10^3 \end{vmatrix} > 0.$$

Hence U is a convex quadratic function and is suitable as a utility function.

The indifference lines are arcs of ellipses. The axes of these ellipses make an angle of 45° with the x_1 and x_2 axes and the center of these ellipses is the vertex of the parabola (2.2).

3. *The n dimensional case.* Professor Charles B. Morrey, Jr. of the Department of Mathematics of the University of California has considered the case of n expenditures and has proved the following theorem:

Theorem. If an Engel Curve is given by $x_i = a_i t^2 + b_i t + c_i$, $i = 1, \ldots, n$, where t is the total income and x_i is the amount spent spent on a particular item then the utility function, U, is of the form $U = K(-2x_1^2 - x_2^2) - Q(x_3, \ldots, x_n) + F$ where K is an arbitrary positive constant, F is any constant, and $Q(x_3, \ldots, x_n)$ is an arbitrary positive definite quadratic form in $(x_3, \ldots, x_n)$.

4. *Summary.* As a general rule the expenditure on an item or on a group of closely related items is a quadratic function of the total expenditure or income.

The situation for two variables, savings and all other expenditures, is discussed in detail. The Engel Curve is a parabola. The utility function, if quadratic, is determined except for an additive constant and a constant multiplier, by this parabolic curve. The indifference curves are arcs of ellipses and the quadratic utility function is convex.

The n dimensional case permits additional elements of arbitrariness in the utility function.

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SYSTEMS OF EQUATIONS, MATRICES AND DETERMINANTS

(Concluded)

Olga Taussky and John Todd

CHAPTER II

The numerical treatment of problems in this field is not entirely straightforward, and requires particular care even when we are handling problems in quite a moderate number of unknowns e.g., for about 10. This whole subject is at present under active investigation in view of the development of high speed automatic digital computing machines, by means of which it is possible to contemplate the solution of problems in which the number of unknowns is of the order of hundreds. These investigations are in various directions. One of these is a re-examination of old methods and includes their history, classification and unification, together with detailed studies of special methods and attempts at the eradication of the superstitions and the justification of the suspicions which are still rife in the rather primitive field of numerical analysis. A second is the devising and examination of newer methods, such as the gradient or finite iteration methods. A third direction is the study of methods appropriate for special systems, such as those which arise by the discretization of differential equations.

We shall show some of the advantages and disadvantages of the classical methods, indicate some of the newer methods, and show some of the difficulties which can arise. This will be done mainly by discussions of special numerical examples.

For up-to-date accounts of this aspect of the subject, reference should be made to Proceedings of a Symposium on Simultaneous Linear Equations and the Determination of Eigenvalues[1]. The article in this by G. E. Forsythe contains a comprehensive bibliography of the first of the two topics. For a more detailed account of methods for the solution of equations and the inverting of matrices, illustrated with numerical examples and directed more towards those who do not have access to high-speed equipment, we refer to the report by L. Fox[2].

II. 1. *SOLUTION OF SYSTEMS OF SIMULTANEOUS EQUATIONS*

Methods of solution can be classified as direct, or as indirect (or iterative). In the first, in theory, we obtain the exact solution

[1]To be published as National Bureau of Standards, Applied Mathematics Series, Vol. 29, 1952.

[2]To appear in National Bureau of Standards Journal of Research, 1952.

after a finite number of steps. In the second we produce an infinite sequence of numbers which converge to the exact solution. In exceptional circumstances the sequence may be stationary, i.e. all terms are equal after a certain stage. There have recently been developed methods which, in theory, are stationary in general. In practice however, because of the fact that our calculations have to be approximate, they are not stationary.

DIRECT METHODS

1. a. *Elimination Method*

This is one of the oldest methods, and, it will be seen, one of the best, at least for general systems. Consider

(1) $R_1 = 12x - 3y + 2z - 96 = 0$ Check sum = -85

(2) $R_2 = -3x - 8y + z - 68 = 0$ Check sum = -78

(3) $R_3 = x + 2y + 6z - 3 = 0$ Check sum = 6

Eliminate x by subtracting 12 times the third equation from the first and by adding 3 times the third equation to the second. We get

(4) $-27y - 70z - 60 = 0$ Check sum = -157

(5) $-2y + 19z - 77 = 0$ Check sum = -60

Eliminate y by multiplying (5) by 27/2 and subtracting from (4). We find

(6) $-\frac{653}{2} z + \frac{1959}{2} = 0$ Check sum = +653

which gives $z = 3$. We now substitute this value of z in (4) (or (5)) to get $y = -10$ and then put these values in (1) (or (2) or (3)) to get $x = 5$.

In this, as in all numerical work, a description of a proposed solution is not complete unless some sort of checking system is incorporated. This should reveal errors as soon as possible after they occur and not at the last stage, if at all. There is available in this case, the following method which can be applied with minor modifications in most of the manipulations in this field. We shall not mention it explicitly again.

We carry an additional column in our work sheet as the sum of the numbers in that row. After writing down our system we compute the numbers -85, -78, 6 labelled "Check sum" above. We perform on these sums the same operations as we do on the equations. Thus we take $-85 - 12 \times 6 = -157$ and $-78 + 3 \times 6 = -60$ and compare these with the sums of the coefficients in (4) and (5) which are $-27 - 70 - 60 = -157$ and $-2 + 19 - 77 = -60$. Proceeding, as a check sum we compute

$-157 - [\frac{27}{2}\times(-60)] = +653$ which agrees with the coefficient sum in (6): $-\frac{653}{2} + \frac{1959}{2} = +653$. A final check might be the substitution of the values of x, y, z in the two equations which were not used in the determination of x.

1. *b*. *Determinantal Solution*

This is often called Cramer's Rule. Applied to our example it gives

$$\frac{x}{\begin{vmatrix} -3 & 2 & -96 \\ -8 & 1 & -68 \\ 2 & 6 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 12 & 2 & -96 \\ -3 & 1 & -68 \\ 1 & 6 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & -3 & -96 \\ -3 & -8 & -68 \\ 1 & 2 & -3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} 12 & -3 & 2 \\ -3 & -8 & 1 \\ 1 & 2 & 6 \end{vmatrix}}$$

i.e.

$$\frac{x}{3265} = \frac{-y}{6530} = \frac{z}{1959} = \frac{-1}{-653}$$

giving $x = 5$, $y = -10$, $z = 3$. Thus the problem is reduced to the evaluation of determinants.

The evaluation of a determinant of order n from its explicit definition becomes rapidly more tedious as n increases. For there are $n!$ terms in the expansion and each involves n factors; we have therefore to carry out about $n! \times n$ multiplications as well as about $n!$ additions. A more efficient method of evaluating determinants is therefore essential. We shall show how it is possible to transform a determinant by means of the transformations of the form discussed in I. 2, which do not alter its value, into one whose expansion contains but one non-zero term. Take the determinant

$$D = \begin{vmatrix} 12 & -3 & 2 \\ -3 & -8 & 1 \\ 1 & 2 & 6 \end{vmatrix}$$

Denote its rows by r_1, r_2, r_3. Subtracting twelve times r_3 from r_1 and adding three times r_3 to r_2 we obtain:

$$D = \begin{vmatrix} 0 & -27 & -70 \\ 0 & -2 & 19 \\ 1 & 2 & 6 \end{vmatrix}$$

Multiplying r_2 by 27/2 and subtracting from r_1 we obtain

$$D = \begin{vmatrix} 0 & 0 & -\frac{653}{2} \\ 0 & -2 & 19 \\ 1 & 2 & 6 \end{vmatrix}$$

The expansion of this determinant contains the single non-zero term

$$-(1)\ (-2)\ (-\frac{653}{2}) = -653.$$

What is the expense of this operation in the general case? Again neglecting additions we see that the first stage contains one division and $n \times (n-1)$ multiplications. Neglecting the divisions we see that we have in all about $\Sigma\, n(n-1) \doteq \frac{1}{3} n^3$ multiplications. This is a considerable improvement on the $n! \times n$ multiplications, even for small values of n.

This method, which we see is essentially equivalent to the elimination method, is recommended for the evaluation of determinants of general type. It also seems reasonable to discard the determinantal method of solution, even if the determinants are evaluated by the method just described, in favor of the elimination method. This method, as applied to determinants often goes by the name of Chió's method of pivotal condensation. The pivots in our example are the numbers 1, -2. There is, of course, plenty of freedom in the choice of the pivots (i.e., the order in which we eliminate the variables). The most efficient choice is being investigated.

1. c. *Finite Iteration Scheme*

The standard method for finding the center of an ellipse

(1) $\qquad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ ab > h^2$

is to solve the system of equations

(2) $\qquad ax + hy + g = 0,\ hx + by + f = 0.$

Its solution gives the center of (1) for any c. We shall now reverse the process: given a system of equations of the form (2) we shall construct the center of the family of ellipses (1). We use the fact that the chord of contact of a pair of parallel tangents passes through the center. Let θ_0 be any direction. Denote by l_0 the chord of contact of the tangents in the direction θ_0. We denote by θ_1 the direction of l_0 and by l_1 the corresponding chord of contact. The intersection of the lines l_0, l_1 is the center. We reach it as follows: Let S_0 be any point. Proceed from S_0 in the direction θ_0 until we meet l_0, at S_1 say. Proceed from S_1 in the direction θ_1 until we meet l_1 at S_2. Then S_2 is the center.

The method can be extended to higher dimensions; for instance, consider the enveloping cylinders, in three directions, of an ellipsoid. The three planes of contact intersect at the center. This can now be approached in three steps.

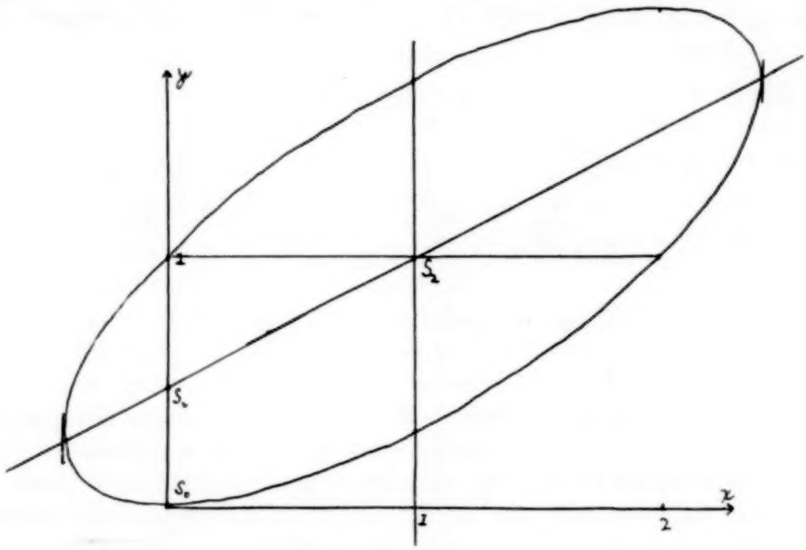


Fig. 1

We shall, for simplicity, in the diagram work out a two-dimensional example. Compare Fig. 1. To solve

$$(3) \qquad x - y = 0, \; -x + 2y - 1 = 0,$$

we determine the center of the ellipse

$$x^2 - 2xy + 2y^2 - 2y = 0.$$

We take $S_0 = (0,0)$ and observe that the *residuals* in (3), i.e. the values of the left hand sides of the two equations for $x = 0$, $y = 0$, are 0, -1. We choose the direction of the residual vector (0,-1), in this case the vertical direction, to be ϑ_0. The chord of contact l_1 is indicated in the diagram. We find the point $S_1 = (0, \frac{1}{2})$. Then we obtain the chord of contact l_1 of tangents parallel to l_0. We then proceed from S_1 along l_0 until we meet l_1 at $S_2 = (1,1)$, the center required.

The general algebraic description of this method for solving $Ax = b$ is the following. We choose any S_0 and observe the residual vector $R_0 = AS_0 - b$ (if $R_0 = 0$ we are finished). We choose $Z_0 = R_0$ and define successively

$$S_{i+1} = S_i + a_i Z_i$$

$$Z_{i+1} = R_i - a_i AZ_i + b_i Z_i, \; R_{i+1} = R_i - a_i AZ_i$$

where the a_i, b_i are certain scalars: a_i is chosen to make S_{i+1} the proper distance from S_i in the direction Z_i and then b_i is chosen to make Z_{i+1} parallel to the planes of contact of all preceding Z_i. We have to take

$$a_i = R_i^2/Z_i AZ_i, \; b_i = R_{i+1}^2/R_i^2.$$

It is found that R_{i+1} is indeed the next residual and that R_n is certainly zero in theory. In practice R_n will be small, and it may be necessary to repeat the process.

INDIRECT OR ITERATIVE METHODS

1. d. *Relaxation*

We now return to the 3-dimensional example of 1.*a*. The solution of the equations (1), (2), (3) is accomplished by choosing values of x, y, z to make the residuals R_1, R_2, R_3 zero theoretically or very small practically. We consider the effect of unit changes on the values of the residuals and obtain an operations table

	R_1	R_2	R_3
x	12	-3	1
y	-3	-8	2
z	2	1	6

The matrix here is the transpose of the matrix of the system. The relaxation process, in its most naive form, starts off with arbitrary values of x, y, z and at each stage liquidates, as nearly as possible, by altering one variable, the largest residual. Considerable virtuosity in the art can be achieved by practice, e.g. instead of altering a single variable several can be altered in a "block-relaxation". This may be suggested internally from the behavior of the residuals or externally from some knowledge of the symmetries in the physical problem which gives rise to the numerical problem.

	x	y	z		R_1	R_2	R_3
	0	0	0		-96	-68	-3
	8				0	-92	5
		-11			33	-4	-17
	-3				-3	5	-20
			3		3	8	-2
		1			0	0	0
Sum	5	-10	3	Check	0	0	0

Starting with a guess 0, 0, 0 we obtain the residuals -96, -68, -3. Looking at the operations table we see that the most economical way to liquidate $R_1 = -96$ is to change x by 8 thereby altering R_1, R_2, R_3 by 96, -24, 8. We obtain the second row above. Now R_2 is the largest residual and we can liquidate it approximately by a change of -11 in y, which means altering R_1, R_2, R_3 by 33, 88, -22. We thus get the third row above. We now reduce R_1 by a change of -3 in x, then R_2 by a change of 3 in z and then R_2 by a change of 1 in y. At this stage, in view of our specially simple example, we get zero residuals. We check our work by adding up the changes in x, y, z to get $x = 5$, $y = -10$, $z = 3$ and check that the residuals are actually 0, 0, 0.

In practical cases we rarely reach an exact solution. What usually happens is that the residuals are reduced by an order of magnitude. If this is not sufficient we change the scale in our residuals and the variables by a factor 10 or 100 and begin again from this first approximation. This can be repeated until a satisfactory solution is obtained.

Thus described, relaxation is a paper and pencil method and we need only work with small integers. The process can be mechanized, but it would be difficult to codify all the tricks of the trade. We point out that the method is particularly effective when the matrix has a dominant main diagonal.

1. e. *"Seidel" Iteration Scheme*

We again take the three-dimensional case. We begin with any guess at the solution, say $x_0 = 1$, $y_0 = 1$, $z_0 = 1$. We obtain first a revised value for x by substituting the old values of y and z in the first equation and solving for x. This gives $x = 8.08$. We then use this value of x and the old value of z in the second equation to obtain a revised value of y: $y_1 = -11.40$. We then use x and y in the third equation to get a revised value of $z = z_1 = 2.95$. We improve this first approximation 8.08, -11.40, 2.95 in the same way. And so on. A few stages of the process are indicated below:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{matrix}12x - 3 + 2 = 96 & : \\ -24.24 - 8y + 1 = 68 & : \\ 8.08 - 22.80 + 6z = 3 & :\end{bmatrix} \quad \begin{bmatrix}8.08\\-11.40\\2.95\end{bmatrix};$$

$$\begin{matrix}12x + 34.20 + 5.90 = 96: \\ -13.98 - 8y + 2.95 = 68 \\ 4.66 - 19.76 + 6z = 3\end{matrix} \quad \begin{bmatrix}4.66\\-9.88\\3.02\end{bmatrix}, \quad \ldots \begin{bmatrix}5.03\\10.01\\3.00\end{bmatrix}$$

This process is convenient both manually and mechanically. Convergence is assured if the matrix is positive definite.

1. f. *Gradient Methods*

We return to the example discussed in 1.c. We observe that the

solution to the (consistent) system

$$x - y = 0, \quad -x + 2y = 1$$

is the point which minimizes the quadratic form

$$\epsilon(x,y) = (x - y)^2 + (-x + 2y - 1)^2$$

the minimum being zero.

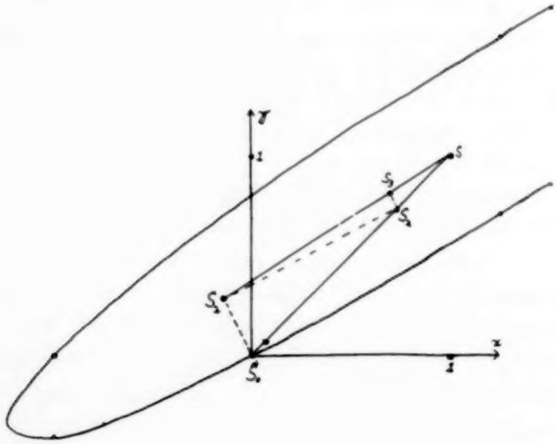


Fig. 2

We take an arbitrary point e.g. $(x_0, y_0) = (0,0)$ as our initial approximation. Compare Fig. 2. We wish to proceed as rapidly as possible to the point where $\epsilon(x,y)$ is minimum. Now $\epsilon(x,y)$ decreases most rapidly at (x_0, y_0) in the direction of the inward normal to that ellipse of the family

$$\epsilon(x,y) = \text{constant} \tag{1}$$

passing through the point. We therefore begin in this direction; it seems reasonable to continue moving in it until we have reached the minimum value of $\epsilon(x,y)$ on this normal: this will occur when the normal becomes a tangent to a member of the family (1). With the choice $(x_0, y_0) = (0,0)$ the direction is $-\left[\left(\frac{\partial \epsilon}{\partial x}\right)_{0,0}, \left(\frac{\partial \epsilon}{\partial y}\right)_{0,0}\right] = (-2,4)$. Our next approximation is

$$x_1 = x_0 - 2r, \quad y_1 = y_0 + 4r$$

where r is such that

$$(x_1 - y_1)^2 + (-x_1 + 2y_1 - 1)^2 - k$$

is a perfect square (in r) for a suitable value of k. We find

$$k = \frac{9}{34},\quad r = \frac{5}{68},\quad x_1 = \frac{-5}{34},\quad y_1 = \frac{10}{34},\quad \epsilon(x_1, y_1) = \frac{9}{34}.$$

We then proceed from (x_1, y_1) in the new direction of steepest descent: $\left(\frac{6}{17}, \frac{3}{17}\right) \equiv (2,1)$.

We find $(x_2, y_2) = \left(\frac{25}{34}, \frac{25}{34}\right)$, $\epsilon(x_2, y_2) = \left(\frac{9}{34}\right)^2$. In a similar way we find

$$(x_3, y_3) = \left(\frac{805}{1156}, \frac{940}{1156}\right),\quad \epsilon(x_3, y_3) = \left(\frac{9}{34}\right)^3.$$

and

$$(x_4, y_4) = \left(\frac{1075}{1158}, \frac{1075}{1158}\right),\quad \epsilon(x_4, y_4) = \left(\frac{9}{34}\right)^4.$$

The approach of (x_n, y_n) to $(1,1)$ is indicated geometrically in the diagram. The points S_0, S_1, ... lie alternately on the lines $y = x$ and $8x - 13y + 5 = 0$.

The question as to whether it would not be advisable to go beyond or stop short of the point of tangency has been investigated in special cases, but no general decision has been made.

II. 2. *INVERSION OF MATRICES*

The solution of the system of equations $Ax = b$ can be obtained by an application of the matrix A^{-1} to the vector b, an operation involving n^2 multiplications. If it is desired, as often happens in practice, to solve the system $Ax = b$ for various values of b (A being unchanged), then it is frequently advantageous to invert the matrix A, once for all.

Many methods are available for this inversion. They can be classified as direct and indirect or iterative as in the previous section. Many of the methods for the solution of systems of equations can be adapted to the more general problem of inversion.

2. a. *Cholesky's Method*

This is essentially based on the elimination method. In its simplest form it applies to a symmetric matrix. We shall discuss the inversion of a matrix A by using its representation as the product of an upper triangular matrix U and its transpose. Thus

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{pmatrix} = U'U = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 + d^2 & bc + de \\ ac & bc + de & c^2 + e^2 + f^2 \end{pmatrix}$$

With this assumption we find in turn $a^2 = 1$, $a = 1$; $ab = 2$, $b = 2$;

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{vmatrix}$$

$ac = 3$, $c = 3$; $b^2 + d^2 = 3$, $d = i$, $bc + de = 4$, $e = 2i$ and $c^2 + e^2 + f^2 = 4$, $f = i$, where we have made arbitrary decisions on the ambiguous signs of c, d and f, which are determined by their squares. It is to be observed that no simultaneous equations have to be solved.

We next observe that

$$A^{-1} = (U'U)^{-1} = (U^{-1})(U')^{-1} = U^{-1}(U^{-1})'$$

and that U^{-1} is easy to find. In fact U^{-1} is also an upper triangular matrix which can be computed from the relation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & i & 2i \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \epsilon \\ 0 & 0 & \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from which we obtain successively $\alpha = 1$, $\beta = 2i$, $\gamma = -i$, $\delta = -i$, $\epsilon = 2i$, $\phi = -i$. Hence we have

$$A^{-1} = \begin{bmatrix} 1 & 2i & -i \\ 0 & -i & 2i \\ 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2i & -i & 0 \\ -i & 2i & -i \end{bmatrix} = \begin{bmatrix} -4 & 4 & -1 \\ 4 & -5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

The method is sometimes called, for obvious reasons, the square root method. We observe that in it we may introduce surds and complex numbers but that these do not appear in the final result – inversion is a strictly rational process.

The second remark is this, that if we apply the Cholesky process to the solution of equations we can reduce our work somewhat. We replace the equation $Ax = b$ by $U'Ux = b$ and this we can replace by $y = Ux$ and $U'y = b$. We therefore solve first for y and then for x. In the case when A is the matrix under consideration and b the vector (14, 20, 23) we can arrange the results of the work compactly as follows:

1	2	3	14	1	0	0	14	1
2	3	4	20	2	i	0	$8i$	2
3	4	4	23	3	$2i$	i	$3i$	3
	A		b		U'		y	x

We also note, that, in addition to linear checks of the type mentioned in 1.2, we can use a quadratic one. We have

$$x'b = x'Ax = x'U'Ux = (Ux)'(Ux) = y'y$$

In the present example we have

$14 \times 1 + 20 \times 2 + 23 \times 3 = 123$ and $14^2 + (8i)^2 + (3i)^2 = 123$.

2.b. *Iteration*

It is well-known that the sequence of real numbers defined by the recurrence relation

$$x_{n+1} = x_n(2 - Nx_n)$$

converges to N^{-1} for suitable x_0, for any real number $N \neq 0$. It can be shown that, for suitable X_0, the sequence of matrices X_n defined by

$$X_{n+1} = X_n(2I - AX_n)$$

converges to A^{-1}, it being assumed that A is non-singular. As an example take the matrix A to be that of 2.a. If we take

$$X_0 = \begin{bmatrix} -3.0 & 4.1 & -0.9 \\ 4.1 & -5.1 & 1.9 \\ -0.9 & 1.9 & -1.1 \end{bmatrix}$$

we obtain

$$X_1 = \begin{bmatrix} -4.26 & 4.14 & -0.86 \\ 4.14 & -5.06 & 1.94 \\ -0.86 & 1.94 & -1.06 \end{bmatrix} \qquad X_2 = \begin{bmatrix} -3.9976 & 3.9864 & -1.0138 \\ 3.9864 & -4.9896 & 2.0104 \\ -1.0136 & 2.0104 & -0.9896 \end{bmatrix}$$

This process is useful for the improvement of approximate inverses.

II. 3. *CHARACTERISTIC ROOTS OF MATRICES*

It has already been remarked that the problem of determining the characteristic roots of a symmetric matrix is essentially the problem of finding the principal axes of a quadric surface. These pure mathematical problems underlie many physical and engineering problems concerned, for instance, with the theory of vibrations.

There are two different problems in this section; according as to whether we wish to determine a few of the dominant characteristic roots (i.e. those with largest moduli), or whether we wish to find all the roots. The first of these problems is the simpler. The following comments refer to the second problem. In the first place, our remarks about the evaluation of determinants indicate that a direct expansion of the determinant, and the assembly of terms into the characteristic polynomial is impractical even if the solution of a polynomial equation could be obtained readily (which is far from being the case). Again, let us assume available a practical method for obtaining a dominant root and a method for obtaining a matrix (of one lower order) having the remaining roots of the original as its characteristic roots. It is tempting to say that repeated application of this process will produce all the characteristic roots. In point of fact, however, the dominant root will only be obtained approximately, and the new matrix will therefore be inaccurate on two counts: due to the inaccuracy of the root and due to the inaccuracies inherent in the numerical calculation involving this approximation to the root. Very soon indeed all significance will be lost. Among the more satisfactory solutions to the

problem is one due to Jacobi: it is described in II. 3. c.

We shall confine our attention to the case of real characteristic roots. This covers the case of symmetric matrices completely.

DOMINANT CHARACTERISTIC VALUES

3.a. *Iteration*

Choose an arbitrary vector $v^{(0)}$ normalized in some way, e.g. to have one of its coordinates unity or to have the sums of the squares of the coordinates unity. Apply the matrix A repeatedly to the vector $v^{(0)}$, expressing each product vector as a scalar multiple of a vector in the chosen normalization. Specifically, if $v^{(i)}$ is normalized then we define $\mu^{(i+1)}$ by the equation

$$Av^{(i)} = \mu^{(i+1)}v^{(i+1)}$$

where $v^{(i+1)}$ is normalized. It can be shown that if A has a single dominant root then these multipliers tend to this value and the (normalized) vectors tend to the corresponding (normalized) characteristic vector.

Let us consider the case

$$A = \begin{pmatrix} 0.2 & 0.9 & 1.32 \\ -11.2 & 22.28 & -10.72 \\ -5.8 & 9.45 & -1.94 \end{pmatrix}$$

and choose $v^{(0)} = (1,0,0)$ and normalize by making the first coordinate unity for simplicity. We obtain the following results:

$$Av^{(0)} = (0.2,-11.2,-518) = \mu^{(1)}v^{(1)} \text{ where } \mu^{(1)} = 0.2,$$

$$v^{(1)} \doteq (1,\ -56,\ -29)$$

$$Av^{(1)} = (-88.48,-948,-478.74) = \mu^{(2)}v^{(2)} \text{ where } \mu^{(2)} = -88.48,$$

$$v^{(2)} = (1,\ 10.7143,\ 5.4107)$$

$$Av^{(2)} = (16.9850,169.5119,84.9534) = \mu^{(3)}v^{(3)} \text{ where } \mu^{(3)} = 16.9850,$$

$$v^{(3)} = (1,\ 9.9801,\ 5.0017)$$

$$Av^{(3)} = (15.7843,\ 157.5384,78.8086) = \mu^{(4)}v^{(4)} \text{ where } \mu^{(4)} = 15.7843,$$

$$v^{(4)} = (1,\ 9.9807,\ 4.9928)$$

$$Av^{(4)} = (15.7731,157.6472,78.8316) = \mu^{(5)}v^{(5)} \text{ where } \mu^{(5)} = 15.7731,$$

$$v^{(5)} = (1,\ 9.9947,\ 4.9979)$$

$$Av^{(5)} = (15.7925,157.9044,\ 78.9540) = \mu^{(6)}v^{(6)} \text{ where } \mu^{(6)} = 15.7925,$$

$$v^{(6)} = (1,\ 9.9987,\ 4.9995).$$

It can be verified that the exact results are $\lambda_1 = 15.8$ and $v_1 = (1, 10, 5)$.

The justification of this process is simple. It is known that, commonly, a matrix A has n different characteristic roots λ_i and n distinct characteristic vectors c_i which are linearly independent and which therefore span the whole space. An arbitrary vector $v^{(0)}$ can be expressed in the form

$$v^{(0)} = \Sigma a_i c_i.$$

Since $Ac_i = \lambda_i c_i$ for $i = 1, 2, \ldots, n$ we have

$$v^{(n)} = A^n v^{(0)} = \Sigma a_i \lambda_i^n c_i$$

and from this, if $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ we have, for sufficiently large n (depending on the separation of the λ's) that

$$v^{(n)} = A^n v^{(0)} \doteqdot a\lambda_1^n c_1$$

From this the statements made above follow: that $v_n^{(n)}$ is approximately a multiple of c_1 and that the ratio of corresponding components of $v^{(n)}$ and $v^{(n-1)}$ is approximately λ_1.

3. b. *Relaxation using Rayleigh's Approximation*

For this method we recall the fact that the dominant characteristic root λ_1 of a symmetric matrix A is the upper bound of the Rayleigh quotient

$$R(x) = \frac{xAx'}{xx'}$$

taken over all vectors x and that this bound is assumed when x is the corresponding characteristic vector, c_1. It is also known that the difference between $R(x)$ and λ_i is comparable with the square of the distance between x and c_1, when x is near enough to c_1. We apply the Relaxation method to systems of equations which are not quite compatible since they involve an approximation to the characteristic root sought.

Take the case when

$$A = \begin{bmatrix} 3.5000 & 0.750 & 1.299 \\ 0.750 & 1.625 & 1.083 \\ 1.299 & 1.083 & 2.875 \end{bmatrix}$$

We guess $v^{(0)} = (1,1,1)$ and estimate the dominant characteristic value

$$u^{(1)} = R(v) = \frac{3.500 + 1.625 + 2.875 + 2[0.750 + 1.299 + 1.083]}{1^2 + 1^2 + 1^2} =$$

$$\frac{14.264}{3} = 4.755$$

We consider the homogeneous system of equations with matrix $A - u^{(1)}I$, i.e.

$$\begin{bmatrix} -1.255 & .750 & 1.299 \\ .750 & -3.130 & 1.083 \\ 1.299 & 1.083 & -1.880 \end{bmatrix}$$

We relax according to the following scheme

$x = 1$	$y = 1$	$z = 1$	Residuals	.794	-1.297	.502
	-0.5			.419	.268	-.040
		-0.1		.289	.160	.148
Sum: $x = 1$	$y = 0.5$	$z = 0.9$	Check:	.2891	.1597	.1485

We now use $v^{(1)} = (1,\ 0.5,\ 0.9)$ as a revised approximation to the characteristic vector and re-estimate the characteristic values $u^{(2)} = R(v^{(1)}) =$

$$\frac{3.500\times1 + 1.625\times0.25 + 2.875\times0.81 + 2[0.750\times1\times0.5 + 1.299\times1\times0.9 + 1.083\times0.5\times0.9]}{1 + 0.25 + 0.81}$$

$$= 4.999$$

We now consider the revised system of equations with matrix $A - u^{(2)}I$: i.e.

$$\begin{bmatrix} -1.499 & .750 & 1.299 \\ 0.750 & -3.374 & 1.083 \\ 1.299 & 1.083 & -2.124 \end{bmatrix}$$

We relax, beginning with $(x, y, z) = v^{(1)}$, thus:

$x = 1$	$y = 0.5$	$z = 0.9$	Residuals	.045	.038	-.071
		-0.3		.006	.006	-.007
		-0.003		.002	.003	-.001
Sum: $x = 1$	$y = 0.5$	$z = 0.867$	Check:	.0020	.0020	-.0010

Thus we can say that 4.999 is an approximation to the dominant characteristic value of the matrix and that the corresponding characteristic vector is (1, 0.5, 0.867).

ALL CHARACTERISTIC ROOTS

3.c. *Jacobi's Method.*

The method to be described is based on the fact that by an orthogonal transformation from variables x,y to variables x', y' which we can describe in the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

we can express the quadratic form

$$ax^2 + 2hxy + by^2$$

as a sum of squares

$$Ax'^2 + By'^2;$$

we have only to choose θ such that $\tan 2\theta = 2h(a-b)^{-1}$.

We take the matrix

$$A = \begin{pmatrix} 2.879 & -0.841 & -0.148 & 0.506 \\ -0.841 & 3.369 & -0.111 & 0.380 \\ -0.148 & -0.111 & 1.216 & -0.740 \\ 0.506 & 0.380 & -0.740 & 3.536 \end{pmatrix}$$

We observe that the largest off-diagonal element is -0.841. In view of the result just quoted it follows that if

$$\tan 2\theta = -2 \times 0.841/(2.879 - 3.369) \text{ and}$$

$$T_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the element in $T_\theta A T'_\theta$ corresponding to -0.841 will be zero, or rather, very small in view of the fact that our calculations are not made exactly. We find, in fact, that

$$A_1 = T_\theta A T'_\theta \begin{pmatrix} 2.250 & 0.001 & -0.186 & 0.636 \\ 0.001 & 4.000 & 0.002 & -0.002 \\ -0.186 & 0.002 & 1.216 & -0.740 \\ 0.636 & -0.002 & -0.740 & 3.536 \end{pmatrix}$$

The largest off-diagonal element is -0.740 and we reduce it by transformation by T_ϕ where

$$T_\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

and where $\tan 2\phi = 2\times(-0.740)/(1.216 - 3.536)$. We find

$$A_2 = T_\phi A\, T'_\phi = \begin{bmatrix} 2.250 & 0.001 & 0.000 & 0.663 \\ 0.001 & 4.000 & 0.001 & -0.002 \\ 0.000 & 0.001 & 1.004 & 0.001 \\ 0.663 & -0.002 & 0.001 & 3.755 \end{bmatrix}$$

We next reduce the element 0.663 by transformation by T_ψ where

$$T_\psi = \begin{bmatrix} \cos\psi & 0 & 0 & \sin\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\psi & 0 & 0 & \cos\psi \end{bmatrix}$$

where $\tan 2\psi = 2\times 0.663/(2.250 - 3.755)$. We find

$$T_\psi A_2 T'_\psi = \begin{bmatrix} 4.005 & -0.002 & 0.000 & -0.001 \\ -0.002 & 4.000 & 0.001 & -0.002 \\ 0.000 & 0.001 & 1.004 & 0.000 \\ -0.001 & -0.002 & 0.000 & 1.999 \end{bmatrix}$$

suggesting that the characteristic roots, which are invariant under the above transformation since the T's are orthogonal, are approximately 4.005, 4.000, 1.004, 1.999.

The proofs that this process is convergent and that it is a practical one are not difficult.

II.4. *ROUNDING-OFF ERRORS*

The question of the effect of rounding errors in calculations is rather difficult but of extreme importance in practice. We shall confine our attention to the case of matrix inversion. What is required in this case are bounds for the difference (in some sense) between the true inverse of a matrix and the result obtained by carrying out a particular process, working always to a fixed number of decimal (or binary) places. These bounds should be readily obtainable and may, of course, be estimates, not necessarily very precise. The bounds may be absolute ones, or probabilistic ones: in the sense that $n\epsilon$ is the absolute error in the sum $\sum_{i=1}^{n} a_i$ due to errors ϵ_i, $|\epsilon_i| \le \epsilon$, in

a_i while the probabilistic error is estimated as $\epsilon\sqrt{n/12}$.

No satisfactory solutions to these problems exist and the results now available are too complicated to discuss here. To underline the importance of these problems we shall discuss a particular numerical example.

Consider the application of some of the methods just described to the case of the solution of the system of equations

$$\left.\begin{aligned} 10x + 7y + 8z + 7w &= 32 \\ 7x + 5y + 6z + 5w &= 23 \\ 8x + 6y + 10z + 9w &= 33 \\ 7x + 5y + 9z + 10w &= 31 \end{aligned}\right\}$$

which has the solution $x = y = z = w = 1$. This example was constructed by T. S. Wilson.

If we eliminate first y and then w and work only to one place of decimals we obtain

$$0.1x + 0.3z = 0.4$$

$$0.4x + 1.2z = 1.6$$

These equations are not independent so that elimination of x also eliminates z.

If we carry three decimals we find

$$0.060\ z = 0.061$$

which gives a poor determination of $z = 1.0??$

If we apply the relaxation process, we can arrive at the following results, where R_1, R_2, R_3, R_4 denote the successive residuals

x	y	z	w	R_1	R_2	R_3	R_4
16.6	-7.2	-2.5	3.1	0.1	-0.1	-0.1	0.1
2.36	0.18	0.65	1.21	0.01	-0.01	-0.01	0.01
1.136	0.918	0.965	1.021	0.001	-0.001	-0.001	0.001

If we apply the Seidel process we obtain the following sequence of approximations:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix};\ \begin{pmatrix}3.20\\0.12\\0.67\\0.20\end{pmatrix};\ \begin{pmatrix}2.44\\0.18\\1.06\\0.35\end{pmatrix};\ \begin{pmatrix}1.98\\0.21\\1.28\\0.46\end{pmatrix};\ \begin{pmatrix}1.70\\0.22\\1.39\\0.55\end{pmatrix};\ \begin{pmatrix}1.55\\0.21\\1.44\\0.61\end{pmatrix};\ \cdots$$

The behavior of this system is blamed on its lack of "condition". Various attempts have been made to measure this lack of condition. The most primitive of these is the smallness of its determinant (in comparison, for example, with the individual terms in its expansion). More satisfactory measures have been introduced. Among these are the M-, N- and P- condition numbers defined by

$M(A) = n\,\mathfrak{M}(A)\mathfrak{M}(A^{-1})$ where $\mathfrak{M} = (C)$ denotes $\max_{i,j} |c_{ij}|$

$N(A) = n^{-1}$ norm A norm A^{-1}

$P(A) = \lambda/\mu$ where λ, μ are the greatest and least of the absolute values of the characteristic roots of A.

When these numbers are large, trouble is to be expected. The actual values of the numbers in the case under discussion are respectively

$$2720, \qquad 752, \qquad 2984$$

which are to be compared with the values

$$5.60, \qquad 3.23, \qquad 9.47$$

for the matrix

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

which is typical of those which arise by the discretization of differential equations.

Estimates for the error in matrix inversion can be given in terms of the condition numbers but it is clear that these numbers cannot be calculated without a knowledge of the inverse (in the first two cases) or of the characteristic roots (in the third).

ERRATA:

Page 13, Sept.-Oct., 1952. The equation

$$A^{-1} = \frac{1}{|A|}((-1)^{i+k}A_{ik})$$

should have read

$$A^{-1} = \frac{1}{|A|}((-1)^{i+k}A_{ik})'.$$

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher,* are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

IDEAL PREPARATION IN MATHEMATICS FOR COLLEGE ENTRANCE*

J. Seidlin

The title is not of my making. But for students and teachers of Mathematics it probably conveys the intended meaning. In the study of ideals or limits we are generally concerned with the behavior of the variable or variables approaching our ideal or limit. Specifically, then, our topic, sheared of symbolism, becomes: "How may grade teachers and high school teachers do a progressively better job of preparing their pupils for optimum learning in courses in college mathematics? That's it; nothing more and, perhaps, nothing less.

On April second I wrote to forty of my friends and colleagues, as follows:

'Dear Professor *X*: On May 3rd I am to tell an assembled multitude of secondary school teachers of mathematics something about "Ideal Preparation in Mathematics for College Entrance". It occurs to me that I should serve "the cause" better if, in addition to my own judgment, I would gather the judgments of the "workers in the field". Would you write me as much or as little as you please, on what you consider the ideal preparation in mathematics for college entrance? May I quote any part or all of what you say?'

I got a wonderful response and I could do worse than just reproduce what they said.

At this point I want to digress a bit, but it is a "relevant digression".

On December 19, 1931, I read a paper before the Mathematics Section of the New York Society of Experimental Research in Education. A year later, December 1932, this paper was reproduced in the Mathematics Teacher under the nasty title of "The Contribution of High School Teaching to Ineffective College Teaching".** Then, as now, I had sought

*Paper read at the Second Annual Meeting of the Association of Mathematics Teachers of New York State, May 3, 1952, at Syracuse, N.Y.

**The Mathematics Teacher, December 1932. (Vol. XXV, Number 8)

and received the help of teachers of college mathematics. Twenty-one years ago I grouped all the criticisms of high school teaching and teachers under twelve headings. The following is a fair sampling:

1. "They (the students) come to us jaded, worn out, *weary* of mathematics".

4. "They do not know the principles underlying the simplest operations." The greater number of our students become involved in the so-called advanced work because

  e. They are generally weak in the application of the laws of exponents (literal, fractional, negative).
  f. They are extravagant in the use of the terms "transpose" and "cancel".

6. "Not one in ten is conscious of the positional scheme of our number system. That is why they experience such difficulty in working with detached coefficients".

11. "They are happiest when a 'proof' (demonstration) is omitted".

Primarily, the high school teacher is not blamed for failing to impart a sufficiently large number of facts to his pupils, but for apparently failing to inspire, or keep alive in, the pupil the more genuine values presumably obtainable through the study of mathematics. The implicit query raised in most instances is: "Of what conceivable value are more or less perfected schemes for imparting facts - facts soon forgotten - if both the foundation and the superstructure of these facts are so fatally impaired?"

What new or different criticisms do we find after a lapse of twenty-one years? Before I answer this question specifically, I want to make a rather significant and encouraging generalization. Today, as contrasted with twenty-one years ago, the college teacher seems to be more keenly aware and appreciative of the high school teacher's enormous burden of trying to meet all the needs of all the children through curricular, co-curricular, and extra-curricular tasks. Thus, while criticisms aimed at the latest product of high school teaching are as plentiful as ever, I detect an attitude of charity, a note of sympathy, and even a modicum of humility.

And now let us get a glimpse at the 1952 edition of "helpful hints" to the producers of freshmen in college mathematics.

1. (and most prominent) Fundamentals and understanding.
  (a) ". . . I wish the young people coming to college really *understood* the fundamental ideas of algebra and trigonometry . . ."
  (b) ". . .the ideally prepared student should *understand* . . . whatever topics he may have studied in high school . . ."
  (c) ". . . in the preparation of students more stress should be laid on *understanding* . . ."
  (d) ". . . unless a student's mathematics has a strong foundation, any super-structure you try to build . . . is a pretty flimsy affair . . ."

(e) ". . . the number of topics covered is not as important as a thorough *understanding* of those that have been covered . . ."

In all but two of the replies "understand" or "understanding" is the key word.

2. Subjects.

Arithmetic and Elementary Algebra lead the list. Plane Geometry is a close third. All of the other secondary school branches of mathematics are mentioned at least once and for our lower and upper limits, respectively, we have the following: "... I think it is relatively unimportant what subject matter is chosen", and "... this would imply that the high schools have given their college matriculants competence in all branches of mathematics *through* the calculus".

However, pretty generally, and stated more or less kindly, we find the plea that analytic geometry and calculus should be judiciously ignored in the high school course of study. Under a less frequently mentioned "hands off" policy we find even advanced algebra and solid geometry, though in the case of these two, recognition is given to the fact that some engineering schools require one or both of them.

All in all, it is not difficult to summarize the dominant position of college professors of mathematics with respect to the prerequisites for college courses. It is not how much mathematics is covered, but how fundamentally well it is learned and retained by the pupils.

3. Boners.

There seems to be no limit to the ingenuity displayed by students in juggling figures or misapplying techniques or perverting principles to suit their own dim and veiled ends. No teaching or teacher, however poor, should be held responsible for this seemingly countless variety of pupils' boners. Still, most of us feel that better teaching would reduce the strain on our sense of humor.

Many of these "intangible errors" are familiar to all of us. Yet every now and then I run across a new one. Why doesn't someone collect them and publish them as a sort of chamber of horrors? Several of my respondents presented me with a few samples, viz.

$$\left.\frac{1\cancel{6}}{\cancel{6}4} = \frac{1\cancel{66}}{\cancel{66}4} = \frac{1\cancel{666}}{\cancel{666}4} = \frac{1}{4}\ \right\}\ \text{Proof of a special kind of cancellation}$$

$$\left.\begin{array}{r|l} & \ \ f \\ \hline x - a & f(x) \\ & \underline{f(x) - f(a)} \\ & \qquad + f(a) \end{array}\ \right\}\ \text{Proof of the Remainder Theorem}$$

$$\frac{2x + \cancel{5}}{\cancel{5}} = 2x$$

$$\frac{1}{a} + \frac{1}{b} = \frac{2}{a + b}$$

and many others.

4. Quotable quotes.

"... I feel that the high schools in New York Stare do a fairly good job of preparation in the field of Mathematics".

"... every attempt should be made to make the student feel that the material studied in one course can be transferred to another course".

"A student who knows the definition of the quotient of two numbers will not wonder why he can't divide by zero".

"Many college freshmen cannot read even a daily newspaper".

"The most frequent and important practical use of trigonometry today is not in the solutions of triangles but in the study of wave motion and other periodic phenomena".

"In view of the appalling state of mathematical ignorance in which students enter *and too often leave* our colleges ..."

"The best definition of a circle that a class of thirty freshmen could produce was 'a circle is when something goes around'".

"Whether the basic difficulty is with the teachers or with the syllabus is something that perhaps could be thrashed out at the Syracuse meeting".

"... they (the students) are so much disturbed and frustrated by the arithmetic and elementary algebra entailed that they have no spirit left to grapple with the problem itself".

"I want them to think. I do not insist on logical or clear or critical thinking; just plain thinking".

"The student should also learn that mathematics is not completely devoid of common sense $(1/32 \div 16 = 2)$".

"Of course we all have many superior students, whose preparation leaves little to be desired".

(From here on I am on my own).

Unquestionably, we have a great many students at all levels of schooling who exhibit the annoying characteristics and practices of the ignorant, the dull, the misfits, the unfits; so that often we feel as Mark Twain did about certain people to whom he referred as: "They are not impossible; they are highly improbable". Our common problem as I understand it, is to reduce both in numbers and intensity these so disturbing characteristics and practices. One obvious way – too obvious – is to plow under the possessors of these characteristics and practices. There are quite a few teachers and administrators who regard "liquidation" as the only solution. That's a dangerous doctrine, since liquida-

tion knows no bounds. It may begin with pupils but it may spread to teachers and administrators. No, these pupils are here to stay.

Also, I am convinced, it was no solution for teachers at one level to snipe at teachers at another level. There is enough poor teaching to go around, and who of us is free from *some* poor teaching! I may be overly optimistic, but from much of what I hear and read I sense a growing realization on the part of teachers at all levels that our job is very much a joint enterprise. Let me paraphrase a variously and more and more frequently expressed injunction to college teachers by college teachers: We must accept our students as we find them. Better we teach them whatever it is they failed to learn than spend an equal amount of time and physical and mental energy on complaining about their high school and grade school teachers and teaching. —

In conclusion, I want to attempt an explanation and offer a suggestion. I know that if I were to ask all the teachers of college mathematics for just *one* bit of advice to teachers of high school mathematics it would be: "Teach for understanding!" I know that if I were to ask all the teachers of high school mathematics what single factor do they stress most in their teaching, it would be: "understanding". How shall we resolve this paradox?

In the kind of learnings which require understanding, there is no way to bypass understanding. *But* – using terminology familiar to all of us – while understanding thus becomes a necessary condition to learning, it is not a sufficient condition. To put it differently, understanding left unattended by drill or practice or application withers away. In this sense understanding is no different from memorization. Somehow our psychologists failed to impress us with this fact. Perhaps they do not know it. Perhaps they "taught" us and we have forgotten.

Be that as it may – and this is the "punch line" – *it is quite possible that so many students give so little evidence of understanding not because they never understood but rather because understandings, unlike old soldiers, do die.*

I realize that even the complete acceptance of this explanation does not solve our problem, but I believe it points the way.

Our best students, and I find substantial agreement on this, – are as good as ever. They understand, they remember, they keep on learning, they are our pride and joy. Our poorest students are probably as poor as ever. I suppose we'll find a better way of dealing with them about the time physicians discover a cure for the "common cold". But it is that large and ever increasing number of students in "the middle" who are really our problem children.

I suggest, I recommend, that for this numerically dominant part of our pre-college and college population we reduce the number of topics in any given subject. (Here the reference is to mathematics, but it could well mean "any given subject") ; that the remaining topics be given a fuller, more leisurely, more meaningful treatment; that a

continuity of topics be established so that no topic is ever "finished"; that every topic be made as rigorous as the training and the background of the pupil permit: no more and no less; that, therefore, the transition from topic to topic, from course to course, and even from high school mathematics to college mathematics (whatever that may mean) may be shock proof because of the smoothness of the road traversed rather than the use of fancy shock absorbers.

It is with some trepidation that I am about to highlight the above suggestions by a reference to a much discussed document, - the proposed six year syllabus.* It is a good syllabus; perhaps the best ever proposed in the State of New York. I approve of it, *as a source book for our best students*. As a syllabus of minimal essentials it insists on a great deal more material than can be taught *meaningfully* to "that large and ever increasing number of students in the middle". So long as we have State Examinations - Regents, or their equivalent - the state syllabus sets the pattern and the pace. The pattern of the new syllabus points in the right direction; the pace, I fear, will not relieve the teachers from the task of coaching for examinations. And the examinations will continue to keep the wolf from the doors of the publishers of Regents Review books.

I wonder how much and what kind of new evidence do we need to convince us that a little learned well is a great deal more useful to the learner than a lot learned (?) poorly; *that "covering ground" is often gratuitous exercise for the teacher and nothing more.***

Give our secondary school teachers half a chance to meet their colleagues at the college level half way. More than ever the *teachers of college mathematics are willing to accept the responsibility for building the superstructure, provided teachers of elementary and secondary school mathematics lay the proper foundation.* Mere numbers of stones or bricks *loosely held together* do not form a proper foundation.

*"The Six-Year High School Mathematics Syllabus for New York State."

**Amen. Ed.

Alfred University

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

MAXIMUM AREA IN A CORNER

The following theorem seems similar to several isoperimetric theorems but doesn't seem to be readily found in the literature. It furnishes the answers to such questions as:

What is the configuration of a simple polygon of maximum area subject to the conditions that two sides meet at a fixed angle and the other $n-2$ sides are prescribed in length only?

I submit it that this apparent gap be filled. The approach is elementary.

THEOREM: Among all the curves of given length whose endpoints A and B lie on two half-lines OL_1 and OL_2 inclined at an angle α (by convention the areas included in the half-plane swept out by OL_1 in rotating through a clockwise angle α are to be considered positive.), *the one which, together with the half-lines OL_1 and OL_2, bounds the maximum area (assuming such a maximumizing curve exists) is a circular arc whose center of curvature is at O if $0 < \alpha < \pi$. If $\pi \leqq \alpha < 2\pi$, the curve is a semicircle.*

Lemma I: The $\triangle AOB$ having the maximum area subject to the conditions $AB = m$ and angle $AOB = \alpha$ is the one in which $OA = OB$.

For of all triangles having a common base and equal opposite angles, the isosceles triangle has the maximum altitude.

Lemma II: Among all the curves of given length s whose endpoints A and B lie at the extremities of a fixed line segment, the one which, together with chord AB, bounds the maximum area is a circular arc.

See e.g., Bolza, Lectures in the Calculus of Variations, ex. 13, p 210.

Proof of the Theorem: If $0 < \alpha \leqq \pi$, the following argument holds.

Suppose there exists a maximizing curve C of length s. Then C must be a circular arc, since otherwise the area could be increased by Lemma II, and the $\triangle AOB$ must be isosceles, since otherwise the area of $\triangle AOB$ could be increased by Lemma I. Thus C is a circular arc with its center on the bisector of angle AOB. We now determine the radius r so that the area will be a maximum.

The area enclosed by the half-lines OL_1 and OL_2 (meeting at an

angle α) and a circle whose center lies on the bisector of α is given by

$$P = Sr/2 - r^2[\sin(S/2r)\sin(\alpha/2 - S/2r)]/[\sin(\alpha/2)]$$

where S is the length of arc C, and r is the radius of curvature of C. For fixed S and α, P is solely a function of r. A necessary condition for a relative maximum is that the derivative of P with respect to r must vanish.

Since

$$\frac{dP}{dr} = \frac{S}{2}[\sin(\alpha/2 - S/2r)/(\sin\alpha/2)][S \cos S/2r - 2r \sin S/2r],$$

the derivative vanishes when

$$(i) \quad \alpha/2 - S/2r = \pm K\pi \quad \text{or} \quad (ii) \quad \tan s/2r = S/2r.$$

The area in (i) is given by

$$P = S^2/(2\alpha \pm 4K\pi) \quad \text{for} \quad K = 0, \ \pm 1 \pm 2 \pm 3 \ldots .$$

Compatible with positive areas, the absolute maximum occurs when $K = 0$.

Consider the function

$$B = \frac{S^2}{8}[\cos(\alpha/2) + \cos(\alpha/2 - S/r)]/[\sin(\alpha/2)] \quad r > 0$$

which coincides with the values of P whenever (ii) is satisfied. It is not hard to see that

$$B \leqq \frac{S^2}{8}[\cos(\alpha/2) + 1]/\sin(\alpha/2) = \frac{S^2}{8}\cot(\alpha/4) < \frac{S^2}{2\alpha}$$

by comparing with the series expansion for $\tan \alpha/4 = \alpha/4 + \ldots > \alpha/4$

$$\text{for} \quad 2\pi > \alpha > 0.$$

Since the absolute maximum of B is less than the absolute maximum of the area given under (i) and the values of P when (ii) is satisfied coincide with those of B. Then all the relative maximums of P under (ii) are less than the absolute maximum under (i). The endpoint maximum occurs when r is infinite which occurs as a special case of (ii) is included therein. The endpoint minimum occurs when $r = 0$. Therefore the absolute maximum occurs when

$$P = \frac{S^2}{2\alpha}$$

whence the center of curvature of C is at O the intersection of the two half-lines OL_1 and OL_2. If $\alpha \leqq \pi$, the above reasoning is valid.

If $\pi < \alpha < 2\pi$, then once the area between the solution C and the chord AB is fixed the maximum occurs when ΔAOB has area zero. This reduces the problem to the famous one of the bullhide and the area enclosed between it and the river. Its well-known solution is a semi-circle.

Summary. If $0 < \alpha < \pi$, the solution is unique. If $\alpha = \pi$, there is an infinite number of solutions. If $\pi < \alpha < 2\pi$, there are two solutions.

Vern Hoggatt

ELLIPSE CONSTRUCTION SHORTCUTS

PROBLEM:

To find shortcuts in the process of constructing ellipses on pictorial drawings from known equal conjugate diameters.

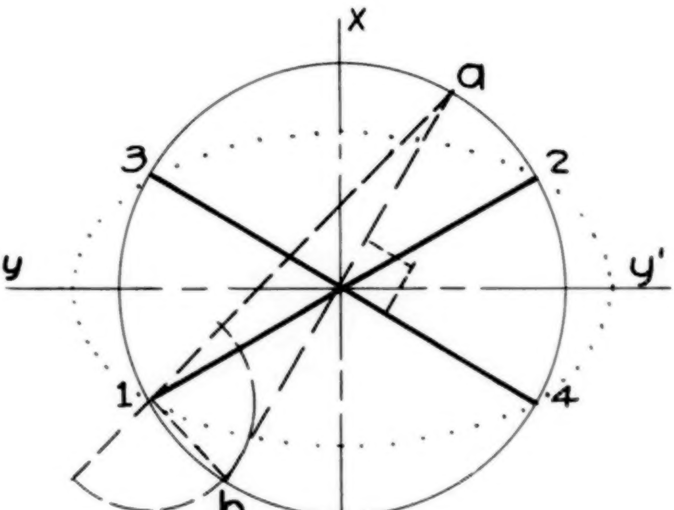


FIG. 1

The basic construction is as shown in Fig. 1. Since the diameters are equal the major and minor axes will bisect the angle between the conjugate diameters and the only problem remaining is to determine their magnitudes. The common construction[1] shown accomplishes this in that the major axis equals $al + bl$ while the minor equals $al - bl$.

[1]Ref. Engineering Drawing, T. E. French, McGraw-Hill; similar construction may be found in Advanced Constructive Geometry, Dowsett, Oxford Univ. Press and Technical Drawing, Giesecke, Mitchell and Spencer, Macmillan.

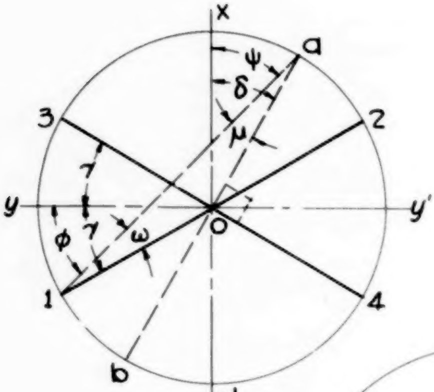


FIG. 2

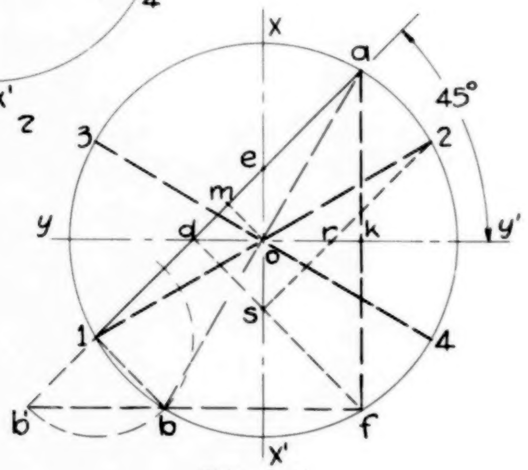


FIG 3

GIVEN:

The pair of equal conjugate diameters 12 and 34, axes xx' and yy', and the lines ab and al, in Fig. 3.

TO PROVE:

$$ad = \frac{al + bl}{2} \text{ and } ae = \frac{al - bl}{2}$$

CONSTRUCTION:

Extend line al to $b'(lb' = lb)$, draw $b'bf$ and akf. Construct fd and om perpendicular to al.

PROOF:

First we prove that the line al will make equal angles (45°) with the major and minor axes, see Fig. 2.

Angles $\gamma = \delta$, since the sides of the angles are perpendicular each to each and both angles are acute.

$$\delta + u = \psi \quad \text{and} \quad \gamma + \omega = \phi.$$

Since the triangle ola is isosceles the angles u and ω are equal and

$$\phi = \psi.$$

$$\phi + \psi + 90° = 180°, \quad \phi + \psi = 90°,$$

$$\phi = \psi = \frac{90°}{2} = 45°.$$

Angle $alb = 90°$ since it is inscribed in a semicircle. Therefore both bl and $b'l$ make 45° with bb' hence bb' is parallel to yy'.

Angle bfa is inscribed in a semicircle and is therefore 90°. $b'f = af$ (opposite sides of an isosceles triangle), therefore the line fd, constructed perpendicular to al, bisects ab'. d is not assumed to be on yy'.

fdk and adk are congruent right triangles since af is perpendicular to yy', also angles at a and f are 45°, therefore dk is perpendicular to af. Therefore dk must lie along yy'.

Construct om perpendicular to al, then $om = dm = em = bl/2$, and $am = al/2$, therefore $am + dm = ad = al/2 + bl/2 = \dfrac{al + bl}{2}$, and $am - em = ae = al/2 - bl/2 = \dfrac{al - bl}{2}$.

APPLICATION:

Since $al + bl$ and $al - bl$ are, by common geometrical construction, the magnitudes of the major and minor axes respectively of the ellipse 1-2-3-4, then ad and ae represent the magnitudes of the semi-major and semi-minor axes.

A line may be drawn thru point 2 parallel to al (45° with yy') and the distances $2r$ and $2s$ measured along this line will be exactly the same as ae and ad. Therefore all other construction may be dispensed with other than this 45° line.

These measurements may then be laid off along axes yy' and xx', and the ellipses constructed by any convenient method.

S. B. Elrod

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Intermediate Algebra. By Paul K. Rees and Fred W. Sparks. McGraw-Hill Book Company, New York, 1951. viii + 328 pages. $3.25.

The authors of this book tell us in the preface that it is intended for a three-semester-hour course for students who have had only one year of high school algebra. It is, therefore, informal in style and offers a great variety of illustrative material that should clarify the discussions of abstract or technical problems.

Several features of the book should appeal to teachers and students alike:

1). Problems have been graded and arranged in groups of four of about the same difficulty. This makes it possible to assign problems of varying complexity by taking every fourth one, or to concentrate on either the simpler ones or the more involved ones as occasion demands. Toward the end of each exercise are more difficult problems that will test the ability of the better students.

2). There are scattered throughout the book numerous notes that point out bits of interesting information (pp. 66, 218, 222, 233), things the student should not do (p. 13), little tricks to help him to avoid errors (pp. 253, 254, 271), how to recognize the easier of two methods of solution (p. 211).

3). Answers are given for the even-numbered problems and for some of the odd-numbered problems so that it is possible for the student to check his own comprehension and accuracy, and thereby gain confidence in his ability.

4). The solution of stated problems is, to many students, the most difficult part of a course in intermediate algebra. The authors have done an unusually fine job in handling problems of this type. They not only explain the general procedure but list specific steps to be followed in analyzing the problem and setting up the necessary equation. In addition, the examples are divided into steps clearly numbered to follow the suggestions already listed (pp. 18-22). The examples

are of many different types including "work", "motion" and "mixture" problems (pp. 88-91). The authors make use of the same kind of step-by-step explanation in statement problems leading to systems of linear equations (pp. 131-135), in problems leading to quadratic equations (pp. 195-197), and in "variation" problems (pp. 243-246). Their explanations and illustrative examples should help the student to master the process of translating stated relationships into corresponding mathematical symbols.

5). Purely mechanical problems are presented with the same detailed explanations, with frequent reference to the proper law or rule that applies in each step of the solution. The explicit directions given by the authors make this an easily understood textbook.

The first nine chapters are built around the equation as the central idea. The discussion leads from the formulas of arithmetic, already familiar to the student (chapter I), through the fundamental operations of algebra (chapter II), special products and factoring (chapter III), to fractions and fractional equations in chapter IV. Next the meaning of function, functional notation and graphical representation of functions are presented (chapter V) as preparation for simultaneous linear equations (chapter VI). Exponents and radicals (chapter VII) are discussed before the introduction of quadratic equations (chapter VIII) and the solution of systems of quadratic equations in two variables (chapter IX). In this chapter (IX) special methods have been avoided and the method of substitution has been emphasized.

Chapter X deals with ratio, proportion, and variation; chapter XI, with logarithms; chapter XII, with progressions; and chapter XIII, with the binomial theorem. The appendix of the textbook includes a four-place table of common logarithms, and a table of powers and roots. The tables add to the usefulness of this well written book.

Vivian Gummo

Advanced Engineering Mathematics. By C. R. Wylie, Jr. McGraw-Hill, N. Y., 1952. $7.50.

Of books on "advanced calculus" there is no end. One wonders, when seeing another one, whether its existence is really justified. The reviewer is inclined to feel that new books on advanced calculus are justified because the increasing importance of the subject in the engineering scene makes it necessary to reach a wider circle of engineering students. Forty years ago when E. B. Wilson wrote one of the first adequate American advanced calculus texts the emphasis was on the needs of theoretical physicists. It was just these physicists who became the engineering analysts and demonstrated the importance of advanced mathematics to engineering analysis. Today real theoretical studies are every day practice in electronics, aeronautics, and other branches of engineering. A working knowledge of advanced calculus

is necessary for the comprehension of many important technical papers. Books on advanced engineering mathematics are necessary. There should be a variety of them to meet a diversity of needs. It appears to be the task of a reviewer to assess these needs and the adequacy of the book for meeting them.

I think that ordinary and partial differential equations, Fourier analysis, Laplace transforms, functions of a complex variable, vector and tensor analysis, including analytic and differential geometry, probability theory, and modern algebraic methods including matrices are all necessary for an understanding of present day engineering analysis. I think that the important part of these subjects can be presented within the compass of an 800 page book. The present book has nothing on algebra and matrices, probability theory, tensor analysis, or differential geometry. The book does treat methods of numerical analysis, a topic which is better taken up as a separate course along with modern digital and analogue computing equipment patterned after the delightful little monograph of Hartree and emphasizing the underlying principles. An "advanced mathematics" book should emphasize principles and illustrate with worked out examples.

What the book does cover it covers well. The emphasis on problem solving is excellent and the worked out examples are instructive. As far as teachability is concerned the reviewer cannot comment. His viewpoint is entirely that of an applied mathematician working in the engineering field. From that standpoint he does feel that the books by Burington and Torrance, Jeffreys and Jeffreys, Margenau and Murphy, Schelkunoff, and Guilleman should also be used for reference purposes although none of these cover probability theory and none of them include all the topics which should be included. A book on advanced engineering mathematics should include many references for further reading. This omission is a common fault and a bad one because it is impossible and undesirable to include all the detail in one volume. An advanced engineering mathematics text should be a starting point not a terminal point in an engineers study of a mathematical technique he intends to apply in his work. He should be able to use such a book as a gateway to a thorough knowledge of the part of the subject he intends to use.

N. G. Parke

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

147. *Proposed by Leon Bankoff, Los Angeles, California.*

In a triangle with integer sides, the side opposite the $120°$ angle is 1729. Find all possible values of the pair of other sides.

148. *Proposed by D. L. MacKay, Manchester Depot, Vermont.*

Upon the sides of triangle ABC the squares $ABDE$, $BCFG$, $ACHL$ are constructed exterior to the triangle. Construct triangle ABC given the points A', B', C' which are the intersections of DE and HL, ED and FG, GF and LH, respectively.

149. *Proposed by L. C. Lay, John Muir College, Pasadena, Calif.*

A has a gambling device so arranged that he always wins 3 times out of every sequence of 5 plays. B suspects that the game is "fixed" and demands that A always wager half of his resources, which B will match with an equal amount. B's funds are assumed to be unlimited.

(1) Show that A always loses over a sequence, regardless of the order of his 3 winning and 2 losing plays.

(2) Determine the best fixed percentage of his resources for A to wager if he wins 3 out of every 5 plays.

(3) If A must always wager 1/2 of his resources, determine to the nearest integer the number of times he must win out of a sequence of one hundred plays in order to break even.

150. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey.*

The lines joining the vertices of a triangle to the internal points of contact of the escribed circles meet in a point. The perpendiculars upon the sides of the triangle from the excenters meet in a point. Show

that these two points together with the orthocenter and the incenter of the triangle are the vertices of a parallelogram.

151. *Proposed by Dewey Duncan, East Los Angeles Junior College.*

In a recent text on *Backgrounds for Secondary Mathematics Teachers* the following statement appears: "Consider the sequence 1, 2, 3, 2 ½, 2, 1 ½, 1 ¾, 2, 2 ¼, 2 1/8, $\cdots$. It has the limit 2, as can be shown by locating the values on a number scale. There is no single formula for this sequence." Refute this last assertion by example.

152. *Proposed by Malcolm Robertson, Rutgers University.*

In finding the area, $A = \pi ab$, of the ellipse $\rho^2 = a^2b^2/(a^2\sin^2\theta + b^2\cos^2\theta)$ a student gets an incorrect answer by proceeding as follows:

$$A = \tfrac{1}{2}\int_0^{2\pi} \rho^2 d\theta = \frac{ab^2}{2}\int_0^{2\pi} \frac{a \sec^2\theta\, d\theta}{b^2 + (a\tan\theta)^2}$$

$$= \frac{ab^2}{2}\left[\frac{1}{b}\arctan\left(\frac{a}{b}\tan\theta\right)\right]_0^{2\pi} = \frac{ab}{2}\left[\arctan 0 - \arctan 0\right] = 0.$$

Detect and explain the source of error.

153. *Proposed by John R. Hatcher, Brown University.*

Prove (without using the exponential function $e^{f(z)}$) that if $f(z) = u + iv$ is entire and $u \neq 0$, then $f(z)$ is a constant.

SOLUTIONS

Sums of Sets of Binomial Coefficients

117. [Nov. 1951] *Proposed by H. D. Grossman, New York, N.Y.*

If n is a positive integer, prove that: a) The maximum difference between any two of the four sums, $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots$, $\binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \cdots$, $\binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \cdots$, $\binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \cdots$, is $2^{[n/2]}$, where $[n/2]$ means the greatest integer $\leq n/2$. b) The maximum difference between any two of the six sums, $\binom{n}{0} + \binom{n}{6} + \binom{n}{12} + \cdots$, $\binom{n}{1} + \binom{n}{7} + \binom{n}{13} + \cdots$, $\cdots$, $\binom{n}{5} + \binom{n}{11} + \binom{n}{17} + \cdots$, is $2 \cdot 3^{(n/2)-1}$ if n is even, or $3^{(n-1)/2}$ if n is odd.

Solution by E. P. Starke, Rutgers University. a) Let S_0, S_1, S_2, S_3, respectively represent the four given sums. From the binomial expansion of $(1 + x)^n$ with $x = i, -1, -i, 1$, we have

$$(1 + i)^n = \binom{n}{0} + i\binom{n}{1} - \binom{n}{2} - i\binom{n}{3} + \cdots,$$

$$(1 - 1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots,$$

$$(1 - i)^n = \binom{n}{0} - i\binom{n}{1} - \binom{n}{2} + i\binom{n}{3} + \cdots,$$

$$(1 + 1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots.$$

Upon multiplying these equations, respectively, by i^a, i^{2a}, i^{3a}, $i^{4a} = 1$, $(a = 1, 2, 3, 4)$ and adding, we obtain

$$i(1 + i)^n - i(1 - i)^n + 2^n = 4S_3, \quad -(1 + i)^n - (1 - i)^n + 2^n = 4S_2,$$

$$-i(1 + i)^n + i(1 - i)^n + 2^n = 4S_1, \quad (1 + i)^n + (1 - i)^n + 2^n = 4S_0,$$

whereupon, putting $D_{ij} = |S_i - S_j|$, we find, after obvious simplifications, that

$$D_{01} = D_{23} = |(1 + i)^{n+1} + (1 - i)^{n+1}|/4, \quad D_{02} = |(1 + i)^n + (1 - i)^n|/2,$$

$$D_{13} = |(1 + i)^{n+2} + (1 - i)^{n+2}|/4, \quad D_{03} = D_{12} = |(1 + i)^{n-1} + (1 - i)^{n-1}|/2.$$

The possible values of the D's are now easily computed. For $n = 2r$, they are 0, 2^{r-1}, 2^r; for $n = 2r + 1$, they are 0, 2^r only. From this, the desired result follows directly.

b) In an analogous fashion we set down the expansions for $(1 + \omega^c)^n$, $(c = 1, 2, \cdots, 6)$, where $\omega = \cos(\pi/3) + i \sin(\pi/3)$. If these are multiplied by ω^a, ω^{2a}, $\cdots$, $\omega^{6a} = 1$ and added, there results

$$\omega^a(1 + \omega)^n + \omega^{2a}(1 + \omega^2)^n + \omega^{4a}(1 + \omega^4)^n + \omega^{5a}(1 + \omega^5)^n + 2^n = 6S_{6-a}$$

for $a = 1, 2, \cdots, 6$. Expressions for the fifteen D_{ij}'s can now be formed and will be found to be of three types:

$$(1) \qquad 6D_{03} = |2(1 + \omega)^n + 2(1 + \omega^5)^n|,$$

$$6D_{14} = |2\omega^2(1 + \omega)^n + 2\omega^4(1 + \omega^5)^n|, \quad 6D_{25} = |2\omega(1 + \omega)^n + 2\omega^4(1 + \omega^5)^n|;$$

(2) $6D_{12} = |\omega^3(1+\omega)^n + \omega^3(1+\omega^5)^n + \omega^{n+1} + \omega^{n+2} + \omega^{5n+4} + \omega^{5n+5}|,$

$$6D_{23} = |\omega^2(1+\omega)^n + \omega^4(1+\omega^5)^n + \omega^{n+5} + \omega^n + \omega^{5n+1} + \omega^{5n}|,$$

and analogous results for D_{34}, D_{45}, D_{05}, D_{01};

(3) $6D_{13} = |\omega^2(1+\omega)^{n+1} + \omega^4(1+\omega^5)^{n+1} + \omega^n + \omega^{n+1} + \omega^{5n} + \omega^{5n+5}|,$

and analogous results for D_{24}, D_{35}, D_{04}, D_{15}, D_{02}. (The relations $\omega^3 = -1$, $\omega^2 - \omega + 1 = 0$ are useful in simplifying.)

It will save space to put H_a for $\omega^a + \omega^{5a}$. If $n = 2r$, the D's take the following forms:

$$D_{03} = 3^{r-1}|H_r|, \quad D_{14} = 3^{r-1}|H_{r+2}|, \quad D_{25} = 3^{r-1}|H_{r+1}|;$$

$$6D_{12} = |3^r H_{r+3} + H_{2r+1} + H_{2r+2}|, \quad 6D_{23} = |3^r H_{r+2} + H_{2r+5} + H_{2r}|, \cdots;$$

$$6D_{13} = |3^r(H_{r+2} + H_{r+3}) + H_{2r} + H_{2r+1}|, \cdots.$$

For $n = 2r + 1$ we have:

$$D_{03} = 3^{r-1}|H_r + H_{r+1}|, \quad D_{14} = 3^{r-1}|H_{r+2} + H_{r+3}|, \quad D_{25} = 3^{r-1}|H_{r+1} + H_{r+2}|;$$

$$6D_{12} = |3^r(H_{r+3} + H_{r+4}) + H_{2r+2} + H_{2r+3}|, \cdots;$$

$$6D_{13} = |3^{r+1}H_{r+3} + H_{2r+1} + H_{2r+2}|, \cdots.$$

Since there are only six distinct powers of ω it is easy to see that, whatever the value of a, H_a must be ± 1 or ± 2. Also $H_a + H_{a+1}$ can be only ± 3 or 0. With these facts in mind and remembering that the D's are all integers, we find the only possible values of the D's. They are,

$$\text{for } n = 2r\text{: } 2\cdot 3^{r-1},\ 3^{r-1},\ (3^r \pm 1)/2,\ (3^{r-1} \pm 1)/2,\ 0;$$

$$\text{for } n = 2r + 1\text{: } 3^r,\ (3^r \pm 1)/2,\ 0.$$

In every case the maximum value is actually attained by one of D_{03}, D_{14}, D_{25}.

If these results are assumed at the outset, it is not difficult to establish them by induction on n. However, the form of D_{ij} depends on the residue of n mod 12, so that induction requires working from n to $n+1$, $n+2$, $\cdots$, $n+12$ before it is complete.

If the binomial coefficients of order n are grouped into five sums

in a manner analogous to the above, the maximum D_{ij} equals the $(n+1)$st Fibonacci number $(1, 1, 2, 3, 5, 8, \cdots; f_{n+1} = f_n + f_{n-1})$. [See *National Mathematics Magazine*, **13**, 292, (March 1939)].

Other references dealing with the sums of sets of the binomial coefficients are: American Mathematical Monthly, **39**, 304, (1932); **45**, 320, (1938); and National Mathematics Magazine, **10**, 165, (February 1936).

Equation Leading to a Fermat Equation

121. [January 1952] *Proposed by Norman Anning, University of Michigan.*

Solve in positive integers, $(x + iy)^3 = x +$ (a pure imaginary). For instance, $(7 + 4i)^3 = 7 + 524i$.

Solution by H. R. Leifer, Pittsburgh, Pa. Let $(x + iy)^3 = x + im$, then $x^3 - 3xy^2 + i(3x^2y - y^3) = x + im$. Equating the real parts, we have $x^3 - 3xy^2 = x$, whereupon either:

(1) $x = 0$, which leads to the trivial non-negative solutions, y equals any positive integer; or

(2) $x^2 - 3y^2 = 1$. This is a Pell (more properly, Fermat) equation, which has an infinity of solutions. Consider

$$\sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

where the successive convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \cdots$$

The numerator of each even convergent gives a value for x and its denominator the corresponding value for y. Thus the first five solutions are (2,1), (7,4), (26,15), (97,56), (362,209). Or, the general solution is given by

$$x = [(2+\sqrt{3})^n + (2-\sqrt{3})^n]/2, \quad y = [(2+\sqrt{3})^n - (2-\sqrt{3})^n]/2\sqrt{3},$$

$$n = (0, 1, 2, \cdots).$$

Also solved by *Leon Bankoff, Los Angeles, Calif; A. L. Epstein, Cambridge, Mass.; M. S. Klamkin, Polytechnic Institute of Brooklyn; F. L. Miksa, Aurora, Ill.; and the proposer.*

Anning gave the recurrence relations $x_n = 2x_{n-1} + 3y_{n-1}$, $y_n = x_{n-1} + 2y_{n-1}$ starting with the obvious solution (1,0). He also noted that $u_n = 4u_{n-1} - u_{n-2}$ applies to both the x's and the y's. *Bankoff* gave $x_n = x_{n-2} + 6y_{n-1}$, $y_n = 2x_{n-1} + y_{n-2}$.

A Circular Cubic

122. [January 1952] *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D.C.*

Show that the envelope of a circle, the square of whose tangent from the origin is equal to the ratio of the abscissa to the ordinate of its center, the center lying on the parabola $x = ay^2$, is a circular cubic, one of whose asymptotes is parallel to the y-axis.

Solution by R. E. Jackson, Student, University of California at Los Angeles. The circle with center at (b, c) has a radius equal to $(b^2 + c^2 - t^2)^{1/2}$ where the length of the tangent, t, from the origin is given by $t^2 = b/c$. Hence the equation of the circle is

$$(x - b)^2 + (y - c)^2 = b^2 + c^2 - b/c.$$

Since $b = ac^2$, the equation of the circle assumes the form

$$f(c) = x^2 - 2ac^2x + y^2 - 2cy + ac = 0.$$

Then

$$df/dc = -4acx - 2y + a = 0.$$

The equation of the envelope of the family of circles is obtained by eliminating c between $f(c)$ and df/dc. Thus

$$8ax(x^2 + y^2) + (2y - a)^2 = 0.$$

The terms of the third degree equated to zero give the lines through the origin parallel to the asymptotes, that is, $x = 0$ and $x = \pm iy$. Hence the cubic is circular. Writing the equation of the cubic in the form $y^2(8ax + 4) - 4ay + 8ax^3 + a^2 = 0$, we see that the asymptote parallel to the y-axis is $x = -1/2a$.

Also solved by *M. S. Klamkin, Polytechnic Institute of Brooklyn; A. Sisk, Maryville, Tenn.; and the proposer.*

Altitude of a Spherical Triangle

123. [January 1952] *Proposed by Joseph Barnett, Jr., Clarksburg, W.Va.*

Theorem: A necessary and sufficient condition that the foot of a perpendicular from the vertex of a spherical triangle to the circle containing the opposite side fall on that side is that the angles adjacent to that side be of the same species.

Solution by the Proposer. Let the side BB' of the spherical triangle $BB'A$ be contained by the great circle $DBD'B'$, with pole P,

and with D (between B' and B) and D' (between B and B') being the intersections of this great circle with the great circle through P and A. Then AD and AD' are the only possible perpendiculars from A to the containing circle.

In triangle BAP, $BA + PA > PB = PD = PA + AD$, so $BA > AD$. Then $90° =$ angle $BDA >$ angle ABD. Similarly, $90° >$ angle $AB'D$. It follows that angles ABD' and $AB'D$ are obtuse. This proves the necessary condition. The converse is true, for if the adjacent angles are of the same species, the perpendicular must lie within the triangle.

Composite Resistances

124. [January 1952] *Proposed by Leo Moser, University of Alberta, Canada.*

Prove that if p and q are positive integers not exceeding the integer n, then it is possible to arrange n or fewer unit resistances to give a combined resistance of p/q.

Solution by the Proposer. We proceed by induction. The theorem is clearly true for $n = 1$ and also for $p = q$.

If $p \neq q$, one of the fractions $p/(q - p)$ or $(p - q)/q$ will have both numerator and denominator positive and less than n. Hence one of these will be obtainable using $n - 1$ or fewer unit resistances. (This is the induction hypothesis.) If the first of these is obtainable, then we get a resistance of p/q by putting it in parallel with a single unit resistance, while if the second is obtainable, we get a resistance of p/q by placing it in series with a unit resistance.

Also solved by *George Baker, Student, California Institute of Technology.*

Problem Related to a Ricatti Equation

125. [January 1952] *Proposed by William Leong, University of California at Berkeley*

Consider the sequence of numbers $\{a_i\}$ where $3a_1 = 1$, $7a_2 = a_1^2$, $11a_3 = 2a_1a_2$, $15a_4 = a_2^2 + 2a_1a_3$, $19a_5 = 2(a_1a_4 + a_2a_3)$, $23a_6 = a_3^2 + 2(a_1a_5 + a_2a_4)$, $\cdots$ (1)

Let $u^4 = \lim_{n\to\infty} a_n/a_{n+1}$. Then show that (a) the number u exists, and (b) u satisfies the equation

$$\sum_{k=1}^{\infty} \frac{(-1)^k u^{4k}}{3\cdot4\cdot7\cdot8\cdots(4k-5)(4k-4)(4k-1)4k} = 0. \tag{2}$$

Solution by the proposer. Consider the Ricatti equation

$$y' = y^2 + x^2 \tag{3}$$

If we set $y = -u'/u$ then $u'' + ux^2 = 0$.

The latter equation has, by proper choice of constants of integration, the series solution

$$u = 1 - c_1x^4 + c_2x^8 - c_3x^{12} + \cdots \tag{4}$$

where $c_1 = \dfrac{1}{3\cdot 4}$, $c_2 = \dfrac{1}{3\cdot 4\cdot 7\cdot 8}$, $\cdots$, $c_n = \dfrac{1}{3\cdot 4\cdot 7\cdot 8\cdots(4k-5)(4k-4)(4k-1)4k}$.

Hence the original equation (3) has a solution

$$y = 4x^3\,\frac{c_1 - 2c_2x^4 + 3c_3x^8 - \cdots}{1 - c_1x^4 + c_2x^8 - c_3x^{12} + \cdots} \tag{5}$$

$$= a_1x^3 + a_2x^7 + a_3x^{11} + a_4x^{15} + \cdots , \tag{6}$$

where the a_i are given by (1).

[The direct division involved in (5) leads to a different set of recurrence formulae from (1):

$$(-1)^n a_n = -4nc_n + c_{n-1}a_1 - c_{n-2}a_2 + \cdots - (-1)^n c_2 a_{n-2} + (-1)^n c_1 a_{n-1} .$$

The formulae (1) are more convenient for calculating the a_i; they were derived by working with (3) instead of the transformed equation.]

We now raise the question of the radius of convergence of series (6). By the ratio test we have that (6) converges for all values of x for which $|x| < \mu$. On the other hand, the series (4) and its differentiated series represent entire functions by the ratio test. Hence it follows from (5) that if $u(x)$ has a zero then μ exists and must be the smallest positive zero of $u(x)$. But it is easy to compute $u(3) = -0.33^-$ and $u(0) = +1$. Hence the statement follows.

The first eleven a_i are: $\dfrac{1}{3}$, $\dfrac{1}{3^2\cdot 7}$, $\dfrac{2}{3^3\cdot 7\cdot 11}$, $\dfrac{13}{3^4\cdot 5\cdot 7^2\cdot 11}$, $\dfrac{46}{3^5\cdot 5\cdot 7^2\cdot 11\cdot 19}$,

$$\frac{15178}{3^6\cdot 5\cdot 7^3\cdot 11^2\cdot 19\cdot 23},\quad \frac{404}{3^7\cdot 5\cdot 7^2\cdot 11^2\cdot 19\cdot 23},\quad \frac{190571}{3^7\cdot 5^2\cdot 7^4\cdot 11^2\cdot 19\cdot 23\cdot 31},$$

$$\frac{5\,858\,822}{3^9\cdot 5^3\cdot 7^4\cdot 11^3\cdot 19\cdot 23\cdot 31},\quad \frac{1\,887\,447\,754}{3^{10}\cdot 5^3\cdot 7^5\cdot 11^3\cdot 13\cdot 19^2\cdot 23\cdot 31},$$

$$\frac{458\,246\,452}{3^{10}\cdot 5^3\cdot 7^5\cdot 11^2\cdot 13\cdot 19^2\cdot 23\cdot 31\cdot 43}.$$

With the aid of a ten-place logarithm table, I have computed the first ten ratios a_n/a_{n+1} and have found that the numbers become stable fairly rapidly. In fact,

$$\sqrt[4]{a_9/a_{10}} = \sqrt[4]{15\,194\,854\,857/943\,723\,877} = 2.00314\,742\cdots,$$

$$\sqrt[4]{a_{10}/a_{11}} = \sqrt[4]{40\,580\,126\,711\,/\,2\,520\,355\,486} = 2.00314\,736\cdots,$$

Also I computed $u(2.003147) = +0.00000\,050\cdots$

$u(2.003148) = -0.00000\,090\cdots.$

Note that at a zero of $u(x)$ the curve of $u(x)$ vs. x has a point of inflection since $u'' = -ux^2$. This fact facilitates the location of the zero.

The series (6) was first derived by James Bernoulli in 1703. Reference is made to a letter to Leibniz (Leibniz: *Gesamellte Werke*, 1855, Vol. 3, p. 75.) This work does not seem to discuss the radius of convergence.

Target Practice

126. [March 1952] *Proposed by George Pate, Gordon Military College, Georgia.*

In firing a rifle at a target from a given distance, suppose that the probability of hitting the bull's-eye is 0.3. What is the smallest number of shots which must be fired in order that the probability of hitting the bull's-eye at least once will be 0.9?

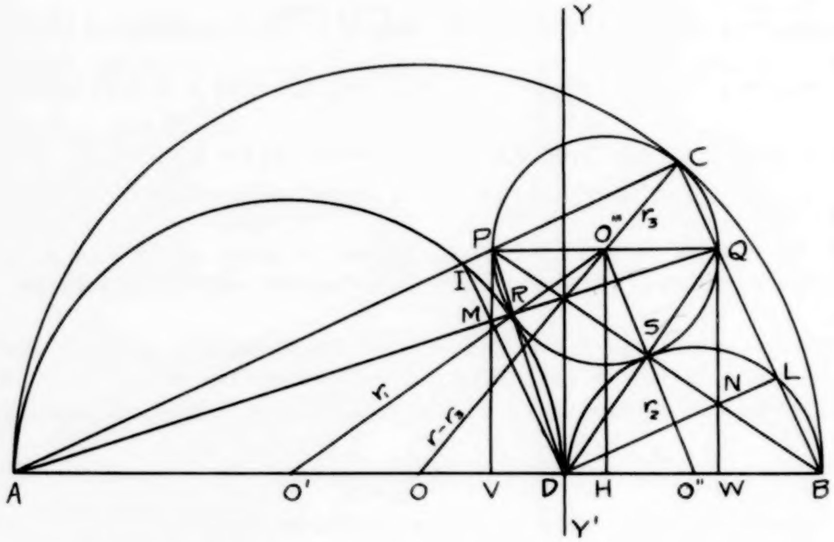
Solution by L. D. Rice, Timken Roller Bearing Co., Canton, Ohio. The probability of not hitting the target with a single shot is $1 - 0.3$ or 0.7. Then $0.7^n = (1 - 0.9) = 0.1$, where n is the number of shots and 0.1 is the probability of missing every time. Hence, $n \log 0.7 = \log 0.1$ and $n = (-1.0000)/(-0.1549) \doteq 6.46$. It follows that seven shots must be fired.

Also solved by *F. F. Dorsey, South Orange, N. J.; J. M. Howell, Los Angeles City College; M. S. Klamkin, Polytechnic Institute of Brooklyn; Joel Levy, University of Michigan; C. D. Smith, University of Alabama; and J. A. Tierney, U. S. Naval Academy.*

Circle Inscribed in the Arbelos

127. [March 1952] *Proposed by M. R. Watson, San Fernando High School, California.*

AB is a diameter of circle (O). The externally tangent circles (O') and (O'') have their centers on AB and are also internally tangent to (O). Find the radius of (O''') which is tangent to (O), (O'), and (O'').



I. *Solution by Howard Eves, Champlain College.* Denote the radii of circles (O), (O'), (O''), (O''') by r, r_1, r_2, r_3 respectively. Take $r_1 \geq r_2$ and let θ denote the non-obtuse angle formed by AB and OO''. Then by the law of cosines,

$$(r_1 + r_3)^2 = (r - r_1)^2 + (r - r_3)^2 + 2(r - r_1)(r - r_3)\cos\theta,$$

$$(r_2 + r_3)^2 = (r - r_2)^2 + (r_1 - r_3)^2 - 2(r - r_2)(r - r_3)\cos\theta.$$

Eliminating $\cos\theta$, we find

$$[(r_1 + r_3)^2 - (r - r_1)^2 - (r - r_3)^2](r - r_2)$$

$$+ [(r_2 + r_3)^2 - (r - r_2)^2 - (r - r_3)^2](r - r_1) = 0.$$

Then, using the fact that $r_1 + r_2 = r$, we obtain

$$r_3 = (r_1 + r_2)r_1r_2/(r_1^2 + r_1r_2 + r_2^2) \text{ or } 2rr_1r_2/(r^2 + r_1^2 + r_2^2).$$

II. *Solution by A. Sisk, Maryville College, Tennessee.* In the triangle $O'O''O'''$, using OO''' as the cevian, by Stewart's Theorem we have

$$(r - r_3)^2(r_1 + r_2) = (r_1 + r_3)^2(r - r_2) + (r_2 + r_3)^2(r - r_1)$$

$$- (r_1 + r_2)(r - r_2)(r - r_1).$$

Solving for r_3, we secure $r_3 = (r_1 + r_2)r_1r_2/(r_1^2 + r_1r_2 + r_2^2)$.

III. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* Applying Heron's formula to triangles $O'O''O'''$, $O'OO'''$, $O''OO'''$ and equating areas, we get

$$\sqrt{(r+r_3)r_3(r-r_1)(r-r_2)} = \sqrt{rr_1r_3(r-r_1-r_3)} + \sqrt{rr_2r_3(r-r_2-r_3)}.$$

Clearing of radicals and using the relationship $r = r_1 + r_2$, we obtain

$$r_3^{\,2}(r^2 - r_1r_2)^2 + 2rr_1r_2r_3(r^2 - r_1r_2) - 3r^2r_1^{\,2}r_2^{\,2} = 0.$$

Thus $r_3 = rr_1r_2/(r^2 - r_1r_2)$ or $r_1r_2(r_1 + r_2)/(r_1^{\,2} + r_1r_2 + r_2^{\,2})$.

IV. *Solution by A. L. Epstein, Cambridge Research Center, Massachusetts.* In a system of coordinates with the origin at the point of tangency of (O') and (O'') and the X-axis coinciding with AB, the centers of the circles will be $O(-r_1 + r_2, 0)$, $O'(-r_1, 0)$, $O''(r_2, 0)$, and $O'''(x, y)$. Then by the distance formula, we have

$$(O'O''')^2 = (x + r_1)^2 + y^2 = (r_1 + r_3)^2,$$

$$(O''O''')^2 = (x - r_2)^2 + y^2 = (r_2 + r_3)^2,$$

$$(OO''')^2 = (x + r_1 - r_2)^2 + y^2 = (r_1 + r_2 - r_3)^2.$$

Now the necessary and sufficient condition that these three equations in $x^2 + y^2$, x, and 1 have a non-trivial solution is that the determinant of their coefficients vanish. That is,

$$\begin{vmatrix} 1 & 2r & (2r_1 + r_3)(-r_3) \\ 1 & -2r_2 & (2r_2 + r_3)(-r_3) \\ 1 & 2(r_1 - r_2) & (2r_1 - r_3)(-2r_2 + r_3) \end{vmatrix} = 0.$$

Whereupon $r_3 = r_1r_2(r_1 + r_2)/(r_1^{\,2} + r_1r_2 + r_2^{\,2})$. There are two tritangent circles with this radius and symmetrical to AB.

V. *Solution by Leon Bankoff, Los Angeles, California.* Each of the two congruent parts into which AB divides this configuration is an *arbelos* with an inscribed circle. By a theorem of Pappus [see R. A. Johnson, Modern Geometry, Houghton-Mifflin Co. (1929), page 117] the perpendicular $O'''H$ from O''' upon AB is equal to $2r_3$. Hence the area of triangle $O'O''O'''$ is $2r_3(r_1 + r_2)/2$. By Heron's formula this area also is $\sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}$. When these expressions for the area are equated, and the equation is solved, we obtain $r_3 = r_1r_2(r_1 + r_2)/(r_1^{\,2} + r_1r_2 + r_2^{\,2})$.

Another approach can be made by using a theorem of Casey [A Sequel to Euclid, Dublin (1884), page 118]: If a variable circle touches two fixed circles, its radius has a constant ratio to the perpendicular from its center on the radical axis. If we consider (O'') and (O''') variable,

$$r_3/AH = r_2/AO'' \quad \text{or} \quad AH = r_3(2r_1 + r_2)/r_2 .$$

If we consider (O') and (O'') variable,

$$r_3/HB = r_1/BO' \quad \text{or} \quad HB = r_3(2r_2 + r_1)/r_1 .$$

Since $AH + HB = AB = 2r$, we have

$$2r = r_3(2r_1 + r_2)/r_2 + r_3(2r_2 + r_1)/r_1$$

whereupon $r_3 = rr_1r_2/(r_1^2 + r_1r_2 + r_2^2)$ as before.

Still another general method follows the specific one of Archimedes given in *OEUVRES D'ARCHIMEDE* (literal translation by F. Peyrard), Francois Buisson, Paris (1807), p. 431. Let PQ be the diameter of O'' parallel to AB; R and S and D respectively be the common points of O'',O' and O''',O'' and O',O''; I and L respectively be the intersections of $CA,(O')$ and $CB,(O'')$; M and N respectively be the intersections of DI,AQ and DL,PB; V and W respectively be the intersections of PM,AB and QN,AB. Triangles $CO''P$ and COA are isosceles and similar so CA passes through P. Likewise, CB passes through Q, QA passes through R, and PB passes through S. Now angles inscribed in semi-circles are right angles so PRD and QSD are straight lines. In triangle APD, AR and DI are altitudes, so PV is an altitude also. Likewise, QW is also perpendicular to AB. It follows from the properties of parallel lines that

$$AD/DB = AM/MQ = AV/VW \quad \text{or} \quad 2r_1/2r_2 = AV/2r_3, \quad \text{and}$$

$$AD/DB = PN/NB = VW/WB \text{ or } 2r_1/2r_2 = 2r_3/WB.$$

Then

$$2r_1 + 2r_2 = AV + VW + WB = 2r_1r_3/r_2 + 2r_3 + 2r_2r_3/r_1.$$

Finally

$$r_3 = (r_1 + r_2)/(r_1/r_2 + 1 + r_2/r_1) \text{ or } (r_1 + r_2)r_1r_2/(r_1^2 + r_1r_2 + r_2^2).$$

A simple construction of (O''') proceeds as follows: Connect C, the extremity of the radius $O''C$ perpendicular to AB, and O'. At D, the point of tangency of (O') and (O''), erect a perpendicular to AB meeting CO' in E. On $O''O'$ lay off $O''F = DE$. Draw FE, and $O''G$ parallel to

FE with G falling on $O'C$. Let K be the foot of the perpendicular from G to AB. With O' as center and radius equal to $O'D + GK$, and with O'' as center and radius equal to $O''D + GK$ describe circles. With the intersection of these circles as center and radius GK describe (O''').

The proof of the construction starts with the similar triangles EDO' and $CO''O'$, in which $ED/DO' = CO''/O''O'$, so $ED = r_1r_2/(r_1 + r_2)$. Then in similar triangles $GO''O'$ and EFO', $GK/O'O'' = ED/O'F$, so

$$GK = [(r_1 + r_2)r_1r_2/(r_1 + r_2)]/[r_1 + r_2 - r_1r_2/(r_1 + r_2)]$$

or

$$r_1r_2(r_1 + r_2)/(r_1{}^2 + r_1r_2 + r_2{}^2).$$

Also solved by *Leon Bankoff* (in a fourth way using Method II), and by *C. D. Smith, University of Alabama.*

Editorial note: This is a special case of the general problem of Apollonius. If three circles with radii r_1, r_2, r_3 touch each other externally, then the radii of the two circles which touch the three circles are

$$r = r_1r_2r_3/[2\sqrt{r_1r_2r_3(r_1 + r_2 + r_3)} \pm (r_1r_2 + r_2r_3 + r_3r_1)]$$

contrary to the statement made in E. W. Hobson, *A Treatise on Plane Trigonometry*, Cambridge, Fourth Edition (1918), page 216, problem 29. If we use the negative sign and impose the condition that $r = r_1 + r_2$, we obtain the result of the current problem.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

"**Q 15.** [Sept. 1950] Find the sum of the squares of the coefficients in the expansion of $(a + b)^n$." *M. S. Klamkin* offers this alternative solution. $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$, and $(1 + 1/x)^n = \binom{n}{0} + \binom{n}{1}/x + \cdots + \binom{n}{n}/x^n$. Multiplying we have that the sum of the squares of the coefficients is the constant term (middle term) of $(1 + x)^{2n}/x^n$. That is, $2n!/(n!)^2$ or $\binom{2n}{n}$.

Q 70. [September 1952]. The last sentence of **A 70**, page 54, should read: "Therefore the maximum value sought is

$$\left[\sum_{n=1}^{N} a_n^2\right]^{N+1}/(N + 1)^{N+1}\prod_{n=1}^{N} a_n \ ."$$

Q 71. Prove that the moment of inertia of a square about a line in the plane of the square and through its center is independent of the orientation of the line. [*Submitted by Leo Moser.*]

Q 72. Evaluate: $\lim_{n\to\infty} 2n/\csc(\pi/n)$. [*Submitted by Charles Salkind.*]

Q 73. Find the sum of the squares of the roots of $3x^4 - 36x^3 + 15x^2 - 1952 = 0$. [*Submitted by D. E. Thoro.*]

Q 74. In Ripley's "New Believe It or Not Book" the following problem is given: "What number if divided by 10 leaves a remainder of 9, divided by 9 leaves a remainder of 8, divided by 8 leaves a remainder of 7, $\cdots$, divided by 2 leaves a remainder of 1." The answer given in that book is 14,622,047,999. Find a smaller solution. [*Submitted by Leo Moser.*]

ANSWERS

A 71. By the symmetry of the square, the moment about such a line will equal the moment about a second such line, perpendicular to it, and in the plane of the square. By the perpendicular axis theorem, the sum of these moments is equal to the moment about a line through the center and perpendicular to the plane of the square. This last moment is clearly independent of the orientation of the original line, so the result follows.

A 72. $2n/\csc(\pi/n) = n\cdot 2\sin(\pi/n)$, which is the perimeter of a regular polygon of n sides inscribed in a unit circle. In the limit this perimeter becomes the circumference of the circle, 2π.

Comment by the Editor. A similar method of evaluation, which also avoids the use of the $(\sin x)/x$ ratio, is: $2n/\csc(\pi/n) = 2n\sin(\pi/n) = 2n\cdot 2\sin(\pi/2n)\cos(\pi/2n) = 2\{2n\cdot 2[\tfrac{1}{2}\sin(\pi/2n)\cos(\pi/2n)]\}$, which is twice the area of a regular polygon of $2n$ sides inscribed in a unit circle. In the limit this area becomes twice the area of the circle, 2π.

A 73. Let a, b, c, d be the roots. From the elementary theory of equations we know that $a+b+c+d = -(-36/3)$ or 12, and that $(ab+ac+ad+bc+bd+cd) = 15/3$ or 5. However, $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$. Hence $a^2+b^2+c^2+d^2 = (12)^2 - 2(5)$ or 134.

A 74. Each division is to leave a remainder of -1. Hence $10! - 1$ or 3,628,799 will do. Also, since the l.c.m. of 1, 2, $\cdots$, 10 is 2520, then 2519 is the smallest positive solution. An even simpler solution is -1.